First Memoir
On some Properties of the Perfect Positive Quadratic Forms

Georges Fedosevich Voronoi

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To God

Preface

George Fedosevich Voronoi was born on 28th April 1868 , in Zhuravka, Poltava Guberniya, Russia, which is now in Ukraine. He died on 20th November 1908 in Warsaw, Poland. Both his master's degree, 1894, on the algebraic integers associated with the roots of an irreducible cubic equation and his doctoral thesis on algorithms for continued fractions were awarded the Bunyakovsky prize by the St. Petersburg Academy of Sciences. But he decided that he wanted to teach at the Warsaw University where he extended work by Zolotarev on algebraic numbers and the geometry of numbers. He met Minkowski in 1904 at an international conference in Heidelberg.

The three papers by Voronoi all appeared in the same journal, the influential and prestigious *Journal für die reine und angewandte Mathematik* [Journal for the pure and applied mathematics]. This journal was a leading mathematical journal during 19th and early 20th centuries when most of its publications appeared either in French or in German. In French it is called *Journal de Crelle* or Crelle's Journal after the name of its founder in 1826 by August Leopold Crelle (1780–1855). Originally Crelle intended the journal to emphasise equally both pure and applied mathematics. But the policy soon changed and it has been dealing solely with pure mathematics since the start of his second and short-lived *Journal für die Bankunst* (1829–1851) to deal the application side.

The main idea is quite simple. It is that space can be partitioned into a set of regions, each surrounding a single point which is sometimes called a nucleus. Every point in a given region is closer to its own nucleus than to any nuclei of other regions. This idea has found so many applications in nature that I think it is as beautiful as the golden ratio $\frac{1+\sqrt{5}}{2}$ is. It has been used to study the forest fire and it has been used to study the structure of the distribution of galaxy. In fact the internal structure of many things has proved to be Voronoi, things in nature as well as man-made ones, for example plant cells and filtering membranes. Looking at a two-dimensional Voronoi structure will remind one of plant or animal cells partitioned by straight walls which result from cosiness of circular cells growing among their neighbours.

Voronoi's lifelong work was in theory of numbers and is divided into three groups, namely algebraic theory of numbers, analytic theory of numbers, and geometry of numbers. His three papers translated here make up two of the planned series of memoirs to apply the principal of Continuous Hermite (Charles Hermite, 1822–1901) parameters to problems of the arithmetical theory of definite and indefinite quadratic forms. He had completed only two of the series when he died in Warsaw, Poland on 20^{th} November 1908 , before the last one appeared in print in 1909. A short obituary written by Kurt Hensel (1861–1941) was included at the end of the paper, which is also included here in translation.

The first paper gives characteristics of complete quadratic forms. In it Voronoi solved the question posed by Hermite on the upper limit of the minima of the positive quadratic forms for a given discriminant of n variables. Zolotarev (Egor Ivanovich Zolotarev, 1847–1878) and Korkin (Aleksandr Nikolaevich Korkin) had given solutions for $n = 4$ and $n = 5$. Voronoi gave an algorithmic solution for any n. He did this with the help of the methods of the geometrical theory of numbers. The present volume gives an English translation of this paper.

The second and the third papers deal with simple parallelepipeds, that is polyhedra with parallelograms as all their faces. He gave the determination of all possible methods of filling an n-dimensional Euclidean space with identical convex non-intersecting polyhedra (parallelepipeds) which have completely contiguous boundaries. A solution of this problem for three-dimensional space had been given by Fedorov (Evgraf Stepanovich Fyodorov, 1853–1919) who was a crystallographer although the proof he gave is said to be incomplete. Minkowski (Hermann Minkowski, 1864–1909) showed in 1896 that the parallelepipeds must have centres of symmetry. He also demonstrated that the number of their boundaries did not exceed $2(2^n - 1)$. Voronoi imposed further the requirement that $n + 1$ parallelepipeds converge at each summit and solved the problem for these conditions completely.

Voronoi's collected works appeared in three volumes under the title of *Sobranie sochineny*, Kiev, 1952–1953. He is sometimes referred to as belonging to the St. Petersburg school of the Theory of Numbers. This must have been the Petersburg Mathematical School sometimes called Chebyshev School or Petersburg School. It was founded by Chebyshev (Pafnuty Lvovich Chebyshev, 1821–1894) and had prominent figures as Grave (Dmitri Aleksandrovich Grave), Krylov (Aleksei Nikolaevich Krylov, 1863–1945), Lyapunov (Aleksandr Mikhailovich Lyapunov, 1857–1918), Markov (Andrei Andreyevich Markov, 1856–1922), Sohotski (Yulian-Karl Vasilievich Sokhotsky, 1842–1927), Steklov (Vladimir Andreevich Steklov, 1864–1926), Korkin, K. A. Posse, and A. V. Vassiliev.

New applications of continuous parameters to the theory of the quadratic form

First Memoir

On some properties of the perfect positive quadratic forms

by

Mr. Georges Voronoï in Warsaw

[*Journal für die reine und angewandte Mathematik*]
[V. 133, p. 97–178, 1908]
[translated by K N Tiyapan]

Introduction

Hermite had introduced in the theory of numbers a new and fruitful principle, namely: being given a set (x) of systems $(x_1, x_2, \ldots, x_n)$ for all the values of $x_1, x_2, \ldots,$ x_n, one associates with the set (x) a set (R) composed of the domains in a manner such that by studying the set (R) one studies at the same time the set (x) .

Hermite has shown † numerous applications of the new principle to the generalisation of continuous fractions, to the study of algebraic units, etc.

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† *Hermite.* Extraits de lettres de M. Ch. Hermite à M. Jacobi sur differents objets de la théorie des nombres. [Excerpts from letters of Mr. Ch. Hermite to Mr. Jacobi on various subjects in the theory of numbers] (This Journal V. 40, p. 261)

Hermite. Sur l'Introduction des variables continues dans la théorie des nombres. [On the introduction of the continuous variables in the theory of numbers] (This Journal V. 41, p. 191)

Hermite. Sur la théorie des formes quadratiques. [On the theory of quadratic forms] (This Journal V. 47, p 313)

The ideas of Hermite have been developed in the works of Mr.'s Zolotareff, Charve, Selling, Minkowski. ‡

I intend to publish a series of Mémoires in which I shall show new applications of the principle of Hermite to the various problems of the arithmetic theory of definite and indefinite quadratic forms.

In this Mémoire, I study the properties of the minimum of positive quadratic forms and of their various representations by systems of integers.

Hermite has discovered an important property of the minimum M of positive quadratic forms $\sum a_{ij}x_ix_j$ in n variables and of the determinant D, namely:

$$M \leq \left(\frac{4}{3}\right)^{\frac{n-1}{2}} \sqrt[n]{D},$$

and he has demonstrated numerous applications of this formula.

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‡ *Zolotareff.* On an indeterminate equation of the third degree (Petersbourg, 1869, in Russian.)

Zolotareff. Theory of complex integers with applications to the integral calculus. (Petersbourg, 1874, in Russian.)

Charve. De la réduction des formes quadratiques ternaires positives et de leur application aux irrationelles de troisième degré. [Of the reduction of positive ternary quadratic forms and of their application to the irrationals of third degree] (Suppl. to V. IX of Annales Scientifiques de l'Ecole Normale Supérieure, 1880)

Selling. Über die binären und ternären quadratischen Formen. [On the binary and ternary quadratic forms] (This Journal, V. 77, p. 143)

Minkowski. Geometrie der Zahlen. [Geometry of numbers] (Leipzig, 1896)

In a letter to Jacobi, Hermite has said §:

"That which precedes sufficiently indicates an infinity of other analogous consequences which, all, will depend on the difficult study of an exact limit of the minimum of any definite form. Thereupon I then form only one conjecture. My first studies in the case of a form in n variables of the determinant D have given me the limit $\left(\frac{4}{3}\right)^{\frac{n-1}{2}} \sqrt[n]{D}$, I am inclined to presume, but without being able to demonstrate that the numerical coefficient $\left(\frac{4}{3}\right)^{\frac{n-1}{2}}$ has to be replaced by $\frac{2}{\sqrt[n]{n+1}}$"

Mr.'s Korkine and Zolotareff has under taken the study of the exact limit of the minimum of positive quadratic forms of the same determinant.

By indicating with $M(a_{ij})$ the minimum and with $D\ (a_{ij})$ the determinant of the form $\sum a_{ij}x_ix_j$, one will have the minimum

$$\mathcal{M}(a_{ij}) = \frac{M(a_{ij})}{\sqrt[n]{D(a_{ij})}}$$

of a positive quadratic form with determinant 1.

By virtue of the theorem of Hermite the function $\mathcal{M}(a_{ij})$ verifies the inequality ¶

$$\mathcal{M}(a_{ij}) \leq \left(\frac{4}{3}\right)^{\frac{n-1}{2}},$$

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§ This Journal. V 40, p. 296

¶ Mr. Minkowski has demonstrated an upper limit of the function $\mathcal{M}(a_{ij})$

$$\mathcal{M}(a_{ij}) \leq n$$

much simpler than that from Hermite.

(*Minkowski.* Über die positiven quadratischen Formen und über kettenbruchähnliche Algorithmen. [On the positive quadratic forms and on continued fraction algorithm] This Journal V. 107, p. 291) therefore it is bounded within the set (f) of all the positive quadratic forms of real coefficients.

Mr.'s Korkine and Zolotareff have demonstrated † that the function $\mathcal{M}(a_{ij})$ possesses many maxima in the set (f) which correspond to the various classes of equivalent positive quadratic forms.

The limit $\frac{2}{\sqrt[n]{n+1}}$ indicated by Hermite in the letter to Jacobi (source cited) is only a maximum value of the function $\mathcal{M}(a_{ij})$.

The binary and ternary positive quadratic forms possess a single maximum which is therefore, in this case, the exact limit of values of the function $\mathcal{M}(a_{ij})$.

Reckoning from the number of variables $n \geq 4$, one meets many maxima of the function $\mathcal{M}(a_{ij})$.

Mr.'s Korkine and Zolotareff have found many values of various maxima of the function $\mathcal{M}(a_{ij})$ which exceed the limit $\frac{2}{\sqrt[n]{n+1}}$ indicated by Hermite, but do not exceed the limit 2.

The study of the exact limit of the minimum of positive quadratic forms of the equal determinant comes down, after Mr.'s Korkine and Zolotareff, to the study of all the various classes of positive quadratic forms to which correspond the maximum values of the function $\mathcal{M}(a_{ij})$.

The maximum maximorum of values of the function $\mathcal{M}(a_{ij})$ is the largest value of the function $\mathcal{M}(a_{ij})$ which presents a numerical function as $\mu(n)$.

Mr.'s Korkine and Zolotareff have determined the following values of the function $\mu(n)$:

$$\mu(2) = \sqrt{\frac{4}{3}},\ \mu(3) = \sqrt[3]{2},\ \mu(4) = \sqrt[4]{4},\ \mu(5) = \sqrt[5]{8},.$$

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† *Korkine* and *Zolotareff.* Sur les formes quadratiques. [On the quadratic forms] Mathematische Annalen, V. VI, p. 366 and V. XI, p. 242

They have called extreme the quadratic forms which yield to the function $\mathcal{M}(a_{ij})$ a maximum value.

The extreme quadratic forms enjoy an important property, namely:

I. Any extreme quadratic form is determined by the value of its minimum and by all the representations of the minimum.

Mr.'s Korkine and Zolotareff have determined all the classes of extreme forms in 2, 3, 4 and 5 vertices.

By studying these extreme forms, I have observed that they are all well defined by the property (I). There is only reckoning from positive forms in six variable which I have encountered positive quadratic forms which enjoyed the property (I) and are not of extreme forms.

I call "perfect" any positive quadratic form which enjoys the property (I).

I demonstrate that the set of all the perfect forms in n variables can be divided into classes the number of which is finite.

All extreme form being, by virtue of the property I, a perfect form, it results in that the function $\mu(n)$ presents the maximum of values of the function $\mathcal{M}(a_{ij})$ which correspond to the various classes of perfect forms.

I have established an algorithm for the search of various perfect forms by introducing a definition of contiguous perfect forms.

To that effect, I make correspond to the set (φ) of all the perfect forms in n variables a set (R) of domains in $\frac{n(n+1)}{2}$ dimensions determined with the help of linear inequalities.

The set (R) of domains in $\frac{n(n+1)}{2}$ dimensions presents a partition of the set (f) of all the positive quadratic forms in n variables.

Each domain R possesses in the set (R) a contiguous domain which is well determined by any one face in - $\frac{n(n+1)}{2} - 1$ dimensions of the domain R.

I demonstrate that the domain R corresponding to the perfect form $\varphi(x_1, x_2, \ldots, x_n)$ being determined by the linear inequalities

$$\sum p_{ij}^{(k)} a_{ij} \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

one will have σ perfect forms defined by the equalities

$$\varphi_k(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho_k \Psi_k(x_1, x_2, \ldots, x_n),$$
$$(k = 1, 2, \ldots, \sigma) \qquad (1)$$

where

$$\Psi_k(x_1, x_2, \ldots, x_n) = \sum p_{ij}^{(k)} x_i x_j,$$

provided that the positive parameter ρ_k $(k = 1, 2, \ldots, \sigma)$ presents the smallest value of the function

$$\frac{\phi(x_1, x_2, \ldots, x_n) - M}{-\Psi(x_1, x_2, \ldots, x_n)}$$

where $\Psi(x_1, x_2, \ldots, x_n) < 0$ and M is minimum of the form $\phi(x_1, x_2, \ldots, x_n)$.

I call "contiguous to the perfect form $\phi(x_1, x_2, \ldots, x_n)$" the perfect forms (1).

Any substitution in integer coefficients and with determinant ±1 belonging to the group g of substitutions which do not change the form ϕ permute only the forms (1). One can, therefore, divide the forms (1) into classes of equivalent forms with the help of substitutions of the group g. By choosing one form in each class, one will have a system of perfect forms contiguous to the perfect form ϕ which can replace the system (1).

By proceeding in this manner, one can obtain a system complete of representatives of various classes of perfect forms.

The corresponding domains will form complete system of representatives of various classes of the set (R).

I have remarked that a similar system

$$R, R_1, R_2, \ldots, R_{\tau-1} \tag{2}$$

of domains of the set (R) can serve towards the reduction of positive quadratic forms.

I call reduced any positive quadratic form belonging to one of the domains (2).

It results from this definition:

I. Any positive quadratic form can be transformed into an equivalent reduced form, with the help of a substitution which presents a product of substitutions belonging to a series of substitutions

$$S_1, S_2, \ldots, S_m$$

which depend only on the choice of the system (2).

II. Two reduced forms can be equivalent only provided that the corresponding substitution belonged to a series of substitutions the number of which is finite.

The weak point of the new method of reduction of positive quadratic forms, demonstrated in this Mémoire, consists in that the number of substitutions which transform into itself the domains of the set (R) or their faces is, in general, very large.

The application of the general theory demonstrated in this Mémoire to the numerical examples will be particularly facilitated if one knew how to solve the following problem:

Being given a group G of substitutions which transform into itself a domain R, one would like to partition this domain into equivalent parts the number of which will be equal to the number of substitutions of the group G and *on condition that the number of faces in* $\frac{n(n+1)}{2} - 1$ *dimensions of domains obtained be the smallest possible.*

I show in this Mémoire the solution of the problem introduced in two cases: $n = 2$ and $n = 3$.

From the number of variable $n \geq 4$, I do not know any practical solution of the problem posed.

First Part
General theory of perfect positive quadratic forms and domains which correspond to them

Definition of perfect quadratic forms.

1

Let

$$\phi(x_1, x_2, \ldots, x_n) = \sum a_{ij} x_i x_j \tag{1}$$

be any positive quadratic form. By indicating with

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}) \tag{2}$$

the various representations of the minimum M of the form $\sum a_{ij} x_i x_j$, one will have the equalities

$$\sum a_{ij} l_{ik} l_{jk} = M, \ \ (k = 1, 2, \ldots, s) \tag{3}$$

One will not consider in the following the two systems

$$(l_{1k}, l_{2k}, \ldots, l_{nk}) \quad \text{and} \quad (-l_{1k}, -l_{2k}, \ldots, -l_{nk}),$$
$$(k = 1, 2, \ldots, s)$$

as different and one will arbitrarily choose one of these systems.

On the ground of the supposition made, one will have the inequality

$$\sum a_{ij}x_ix_j > M$$

provided that a system $(x_1, x_2, \ldots, x_n)$ of integer values of variables $x_1, x_2, \ldots, x_n$ did not belong to the series (2), excluding the system $x_1 = 0, x_2 = 0, \ldots, x_n = 0$.

By considering the equalities (3) as the equations which serve to determine $\frac{n(n+1)}{2}$ coefficients of the quadratic form $\sum a_{ij}x_ix_j$, one will have only two cases to examine:

1.) there exist a finite number of solutions of equations (3),

2.) the equations (3) admit only a single system of solutions.

2

Let us examine the first case,

Let us suppose that there exists an infinite number of solutions of equations (3).

One will find in this case an infinite number of values of parameters

$$p_{ij} = p_{ji}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

verifying the equations

$$\sum p_{ij}l_{ik}l_{jk} = 0, \ (k = 1, 2, \ldots, s) \tag{4}$$

the values $p_{ij} = 0$, $i = 1, 2, \ldots, n; j = 1, 2, \ldots, n$ being excluded.

By indicating

$$\Psi(x_1, x_2, \ldots, x_n) = \sum p_{ij}x_ix_j,$$

let us consider the set of positive quadratic forms determined by the equality

$$f(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n), \tag{5}$$

the parameter ρ being arbitrary.

For a quadratic form determined by the equality (5) to be positive, it is necessary and sufficient that the corresponding value of the parameter ρ be continuous in a certain interval

$$-R' < \rho < R.$$

It can turn out that $R = +\infty$, in this case the lower limit $-R'$ will be finite. By replacing in the equality (5) the form $\Psi(x_1, x_2, \ldots, x_n)$ by the form $-\Psi(x_1, x_2, \ldots, x_n)$, that which is permitted by virtue of (4), one will have the interval

$$-R < \rho < R',$$

therefore one can suppose that the upper limit R is finite.

The corresponding quadratic form, determined by the equality

$$f(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + R\Psi(x_1, x_2, \ldots, x_n),$$

will not be positive, but it will not have negative values either; one concludes that least for a system $(\xi_1, \xi_2, \ldots, \xi_n)$ of real values of variables $x_1, x_2, \ldots, x_n$ the form $f(x_1, x_2, \ldots, x_n)$ attains in its value the smallest which is zero, and it follows that the system $(\xi_1, \xi_2, \ldots, \xi_n)$ verifies the equation

$$\frac{\partial f}{\partial \xi_i} = \frac{\partial \varphi}{\partial \xi_i} + R\frac{\partial \Psi}{\partial \xi_i} = 0. \quad (i = 1, 2, \ldots, n)$$

By eliminating from these equations $\xi_1, \xi_2, \ldots, \xi_n$ one obtains the equation

$$D(R) = \begin{vmatrix} a_{11} + RP_{11}, & a_{12} + RP_{12}, & \ldots, & a_{1n} + RP_{1n} \\ a_{21} + RP_{21}, & a_{22} + RP_{22}, & \ldots, & a_{2n} + RP_{2n} \\ & \ldots & & \\ a_{n1} + RP_{n1}, & a_{n2} + RP_{n2}, & \ldots, & a_{nn} + RP_{nn} \end{vmatrix} = 0$$

The smallest positive root of this equation presents the value of R searched for.

3

Let us examine the set (f) of positive quadratic forms determined by the equality (5) with condition

$$0 < \rho < R. \tag{6}$$

Theorem. *To the set (f) belongs a quadratic form $\varphi_1(x_1, x_2, \ldots, x_n)$ which is well determined by the following conditions:*

1. all the representations of the minimum M of the form $\varphi(x_1, x_2, \ldots, x_n)$ are also representations of the minimum M of the form $\varphi_1(x_1, x_2, \ldots, x_n)$,

2. the form $\varphi_1(x_1, x_2, \ldots, x_n)$ moreover possesses at least another representation of the minimum M.

Let us indicate by $M(\rho)$ the minimum and by $D(\rho)$ the determinant of the quadratic form $f(x_1, x_2, \ldots, x_n)$ defined by the equality (5) with condition (6).

By virtue fo the theorem by Hermite, one will have the inequality

$$M(\rho) \leq \mu(n) \sqrt[n]{D(\rho)}. \tag{7}$$

We have demonstrated that $D(R) = 0$, it results in that a value of the parameter ρ can be chosen in the interval (6) such that the inequality

$$\mu(n) \sqrt[n]{D(\rho)} < M$$

holds. One will have, because of (7),

$$M(\rho) < M. \tag{8}$$

Let us indicate by $(l_1, l_2, \ldots, l_n)$ a representation of the minimum $M(\rho)$ of the form $f(x_1, x_2, \ldots, x_n)$ verifying the inequality (8).

One will have

$$\varphi(l_1, l_2, \ldots, l_n) + \rho\Psi(l_1, l_2, \ldots, l_n) < M, \tag{9}$$

and as a result

$$\varphi(l_1, l_2, \ldots, l_n) > M \text{ and } \Psi(l_1, l_2, \ldots, l_n) < 0. \tag{10}$$

This posed, let us find the smallest value of the function

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{-\Psi(x_1, x_2, \ldots, x_n)} \tag{11}$$

determined with condition

$$\Psi(x_1, x_2, \ldots, x_n) \leq 0. \tag{12}$$

To that effect, let us examine the inequality

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{-\Psi(x_1, x_2, \ldots, x_n)} \leq \frac{\varphi(l_1, l_2, \ldots, l_n) - M}{-\Psi(l_1, l_2, \ldots, l_n)}.$$

By virtue of (9), (10) and (12), one will have

$$\varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n) < M.$$

The quadratic form $\varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n)$ being positive, there exists only a limited number of integer values of $x_1, x_2, \ldots, x_n$ verifying this inequality. Among these systems are found all the systems which give back to the function (11) the smallest value determined with condition (12).

Let us indicate by

$$(l'_1, l'_2, \ldots, l'_n), (l''_1, l''_2, \ldots, l''_n), \ldots, \left(l_1^{(r)}, l_2^{(r)}, \ldots, l_n^{(r)}\right)$$

all the representations of the positive minimum ρ_1 of the function (11).

By declaring

$$\varphi_1(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho_1\Psi(x_1, x_2, \ldots, x_n),$$

one obtains the positive quadratic form $\varphi_1(x_1, x_2, \ldots, x_n)$ the minimum M of which is represented by the systems (2) and (13), this is that which one will demonstrate without trouble.

4

With the help of the procedure previously shown, one will determine a series of positive quadratic forms

$$\varphi, \varphi_1, \varphi_2, \ldots \tag{14}$$

which enjoy the following property: by indicating with s_k the number of representations of the minimum of the form $\varphi_k (k = 1, 2, \ldots)$, one will have the inequalities

$$s < s_1 < s_2 < \cdots \tag{15}$$

A similar series of positive quadratic forms of n variables can not be extended indefinitely, this is that which we will demonstrate with the help of the following lemma.

Lemma. *The number of various representations of the minimum of a positive quadratic form in n variables does not exceed* $2^n - 1$.

Let us indicate by $(l_1, l_2, \ldots, l_n)$ and $(l'_1, l'_2, \ldots, l'_n)$ any two representations of the minimum M of the positive quadratic form $\sum a_{ij} x_i x_j$.

Let us suppose that by declaring

$$l'_i = l_i + 2t_i, \ \ (i = 1, 2, \ldots, n) \tag{16}$$

the number $t_1, t_2, \ldots, t_n$ would be integer.

As

$$\sum a_{ij} l'_i l'_j = M \quad \text{and} \quad \sum a_{ij} l_i l_j = M,$$

by virtue of (16), it becomes

$$\sum a_{ij} l_i t_j + \sum a_{ij} t_i t_j = 0$$

One will present this equality under the form

$$\sum a_{ij}(l_i+t_i)(l_j+t_j)+\sum a_{ij}t_it_j=\sum a_{ij}l_il_j. \qquad (17)$$

By noticing that

$$\sum a_{ij}t_it_j \geq \sum a_{ij}l_il_j,$$

one finds, by virtue of (17),

$$\sum a_{ij}(l_i+t_i)(l_j+t_j) \leq 0,$$

therefore it is necessary that

$$\sum a_{ij}(l_i+t_i)(l_j+t_j)=0,$$

and consequently

$$l_i+t_i=0. \quad (i=1,2,\ldots,n)$$

Because of (16), one obtains

$$l_i'=-l_i. \quad (i=1,2,\ldots,n)$$

This posed, let us divide the set (X) of all the systems $(x_1,x_2,\ldots,x_n)$ of integer values of $x_1,x_2,\ldots,x_n$ into 2^n classes, with regard to the modulo 2.

We have demonstrated that two different representations of the minimum M of the form $\sum a_{ij}x_ix_j$ will not belong to the same class; neither will any representation of the minimum M belong to the class made up of systems $(x_1,x_2,\ldots,x_n)$ satisfying the condition

$$x_i \equiv 0 (\mathrm{mod} 2), \quad (i=1,2,\ldots,n)$$

therefore the number of various representations of the minimum of a positive quadratic form can not be greater than 2^n-1.

5

We have demonstrated that the series (14) of positive quadratic forms satisfying the condition (15) can not be extended indefinitely, therefore the series (14) will be terminated by a form φ_k which enjoys the following property: the form φ_k is determined by the representations of its minimum.

Definition. *One will call perfect any positive quadratic form which is determined by the representations of its minimum.*

Let us suppose that the form (1) be perfect, one will have in this case only a single system of solutions of equations (3).

On the ground of the supposition made, the equations

$$\sum p_{ij}l_{ik}l_{jk}=0, \quad (k=1,2,\dots,s)$$

admit only a single system of solutions

$$p_{ij}=p_{ji}=0. \quad (i=1,2,\dots,n; j=1,2,\dots,n)$$

By effecting the solution of equations (3), one obtains the equalities

$$a_{ij}=\alpha_{ij}M, \quad (i=1,2,\dots,n; j=1,2,\dots,n)$$

where the coefficients α_{ij} are rational.

It results in that the perfect form $\frac{\varphi}{M}$ is of rational coefficients. In the following one will not consider as different the perfect forms of proportional coefficients.

6 *Fundamental properties of perfect quadratic forms.*

Let

$$\varphi(x_1,x_2,\dots,n)=\sum a_{ij}x_ix_j$$

be a perfect quadratic form. Let us suppose that all the different representations of the minimum of the perfect form φ make up the series

$$(l_{11},l_{21},\dots,l_{n1}),(l_{12},l_{22},\dots,l_{n2}),\dots,(l_{1s},l_{2s},\dots,l_{ns}). \quad (1)$$

By choosing any n systems in this series, let us examine the determinant

$$\begin{vmatrix} l_{11}, & l_{12}, & \dots,l_{1n} \\ l_{21}, & l_{22}, & \dots,l_{2n} \\ & \dots & \\ l_{n1}, & l_{n2}, & \dots,l_{nn} \end{vmatrix}=\pm\omega. \quad (2)$$

All the determinants that one can form this way can not cancel each other out. By supposing the contrary, one will have s equations of the form

$$l_{ik} = \sum_{r=1}^{n-1} l_{ir} u_r^{(k)}, \ \ (i = 1, 2, \dots, n; k = 1, 2, \dots, s) \qquad (3)$$

One will choose a system of $\frac{n(n+1)}{2}$ parameters $p_{ij} = p_{ji}$ verifying $\frac{n(n+1)}{2}$ equations

$$\sum p_{ij} l_{ir} l_{jt} = 0, \qquad (r = 1, 2, \dots, n-1; t = 1, 2, \dots, n-1)$$

and by virtue of (3), one will have

$$\sum p_{ij} l_{ik} l_{jk} = 0, \qquad (k = 1, 2, \dots, s)$$

which is impossible.

The numerical value ω of the determinant (2) can not exceed a fixed limit. To demonstrate this, let us effect a transformation of the perfect form φ with the help of a substitution

$$x_i = \sum_{r=1}^{n} l_{ir} x'_r; \ \ (i = 1, 2, \dots, n) \qquad (4)$$

one will obtain a form

$$\varphi'(x'_1, x'_2, \dots, x'_n) = \sum a'_{ij} x'_i x'_j,$$

where

$$a'_{ii} = M. \ \ (i = 1, 2, \dots, n) \qquad (5)$$

By indicating with D' the determinant of the form φ', one will have the inequality

$$a'_{11} a'_{22} \cdots a'_{nn} \geq D',$$

by virtue of the known property of positive quadratic forms.

Considering (5), one obtains

$$M^n \geq D'. \qquad (6)$$

By indicating with D the determinant of the form φ, one will have, because of (2) and (4),

$$D' = D\omega^2,$$

therefore the inequality (6) reduces to the one here:

$$D\omega^2 \leq M^n.$$

By virtue of the theorem by Hermite, one has the inequality

$$M \leq \mu(n)\sqrt[n]{D};$$

it follows that

$$\omega \leq [\mu(n)]^{\frac{n}{2}}.^{\dagger} \tag{7}$$

7

Any perfect form will obviously be transformed into a form, also perfect, with the help of all linear substitution of integer coefficients and of determinant ±1.

One concludes this that there exists a finite a finite number of equivalent perfect forms.

The set (φ) of all the perfect forms in n variables can be divided into different classes provided that each class be made up of all the equivalent perfect forms.

Theorem. The number of different classes of perfect forms in n variables is finite.

Let us indicate by

$$\lambda_k = l_{1k}x_1 + l_{2k}x_2 + \ldots + l_{nk}x_n \quad (k = 1, 2, \ldots, s)$$

s linear forms

$$\lambda_1, \lambda_2, \ldots, \lambda_s \tag{8}$$

---

† See the Mémoire of Mr.'s Korkine and Zolotareff *sur les formes quadratiques positives.* (Mathematische Annalen V. XI, p. 256)

which correspond to the systems (1) of representations of the minimum of the form Ψ.

One establishes this way a uniform correspondence between a perfect form φ and the system (8) of linear forms.

Let us suppose that one had transformed the perfected form φ with the help of a substitution S by integer coefficients and with determinant ±1, one will obtain an equivalent perfect form φ'. Let us indicate by

$$\lambda'_1, \lambda'_2, \ldots, \lambda'_s \tag{9}$$

the corresponding system of linear forms.

One will easily demonstrate that the substitution T, adjoint to the substitution S §, will transform the system (8) into a system (9).

One concludes that a certain reduction of perfect forms can be effected with the help of the reduction of corresponding systems of linear forms.

The reduction of the system (8) comes down, by virtue of (7), to the reduction of any n linear forms

$$\lambda_1, \lambda_2, \ldots, \lambda_n \tag{10}$$

belonging to the system (8) and with determinant $\pm\omega$ which does not cancel each other out.

---

§ The substitution S being defined by the equalities

$$x_i = \sum_{k=1}^{n} \alpha_{ik} x'_k, \quad (i = 1, 2, \ldots, n)$$

one calls "substitution adjoint to S" the substitution T which is determined by the equalities

$$\sum_{k=1}^{n} \alpha_{ik} x_k = x'_i. \quad (i = 1, 2, \ldots, n)$$

One will determine with the help of the known method a substitution T which will transform the linear forms (10) of integer coefficients into linear forms,

$$\lambda'_1, \lambda'_2, \ldots, \lambda'_n \tag{11}$$

satisfying to the following conditions

$$\begin{cases} \lambda'_k = p_{k,k}x' + p_{k+1,k}x'_{k+1} + \ldots + p_{n,k}x'_n, \quad (k = 1, 2, \ldots, n) \\ p_{11}p_{22}\cdots p_{nn} = \omega \text{ and } p_{kk} > 0, \quad (k = 1, 2, \ldots, n) \\ 0 \le p_{k+i,k} < p_{kk}. \quad (i = 1, 2, \ldots, n-k; k = 1, 2, \ldots, n) \end{cases}$$

The coefficients of forms (11) being integers, as a result they do not exceed fixed limits.

The substitution T will transform the system (8) into a system

$$\lambda'_1, \lambda'_2, \ldots, \lambda'_s \tag{12}$$

of linear forms. By examining successively the determinants of forms

$$(\lambda'_k, \lambda'_2, \ldots, \lambda'_n), (\lambda'_1, \lambda'_k, \ldots, \lambda'_n), \ldots, (\lambda'_1, \lambda'_2, \ldots, \lambda'_k),$$

$$(k = n+1, n+2, \ldots, s)$$

one will demonstrate that the numerical values of coefficients of all the linear forms (12) do not exceed fixed limits.

The number of similar systems of linear forms in integer coefficients being limited, it results in that the number of different classes of perfect forms is also limited.

On the domains determined with the help of linear inequalities

8

We have seen in Number 7 that the study of perfect forms can be brought back to the study of certain systems of linear forms.

One will acquire a new basis to these studies by making correspond to each perfect quadratic form in n variables a domain in $\frac{n(n+1)}{2}$ dimensions determined with the help of linear inequalities.

One will address first the general problem by studying the properties of domains determined with the help of linear inequalities. ‡

Let us consider a system of linear inequalities

$$p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

in any real coefficients.

One will call point (x) any system $(x_1, x_2, \ldots, x_m)$ of real values of variables $x_1, x_2, \ldots, x_m$ and one will indicate

$$y_k(x) = p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m. \quad (k = 1, 2, \ldots, \sigma)$$

One will call "domain" the set R of points verifying the inequalities

$$y_k(x) \geq 0. \ (k = 1, 2, \ldots, \sigma) \tag{1}$$

Let us suppose that to the domain R belonged to points verifying the inequalities

$$y_k(x) > 0, \quad (k = 1, 2, \ldots, \sigma)$$

one will call such points interior to the domain R, and the domain R will be said to be of m dimensions.

It can be the case that the domain R does not possess interior points. One will demonstrate in this case all the points belonging to the domain R verify at least one equation

$$y_k(x) = 0,$$

the indice h being a value $1, 2, \ldots, \sigma$.

It is important to have a criteria with the help of which one could recognise whether a domain determined by the help of inequalities (1) will be in m dimensions or not.

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‡ See: *Minkowski.* Geometrie der Zahlen [Geometry of the numbers], No. 19, p. 39.

Fundamental principle. *For a domain determined with the help of inequalities (1) to be of m dimensions, it is necessary and sufficient that the equation*

$$\sum_{k=1}^{\sigma} \rho_k y_k(x) = 0 \tag{Ξ}$$

did not reduce into an identity so long as the parameters $\rho_2, \rho_2, \ldots, \rho_\sigma$ *are positive or zero, the values* $\rho_1 = 0, \rho_2 = 0, \ldots, \rho_\sigma = 0$ *being excluded.*

The principle introduced, considered from a certain point of view, is evident, but one arrive at the rigorous demonstration of this principle only with the help of the in depth study of domains determined with the help of linear inequalities.

For more simplicity, one will examine in that which follows only domains satisfying the following conditions: the equations

$$y_k(x) = 0 \;\; (k = 1, 2, \ldots, \sigma) \tag{2}$$

can not be verified by any point, the point $x_1 = 0, x_2 = 0, \ldots, x_m = 0$ being excluded.

It is easy to demonstrate that the general case will always come down to the case examined.

9

Definition. *One will call edge of the domain R determined with the help of inequalities (1) the set of points belonging to the domain R and verifying the equations*

$$y_k(x) = 0, \quad (k = 1, 2, \ldots, r \text{ where } r < \sigma)$$

provided that these equations defined the values of $x_1, x_2, \ldots, x_m$ *to an immediate common factor.*

By indicating with $(\xi_1, \xi_2, \ldots, \xi_m)$ a point of the edge considered, one will determine all the points of the edge with the help of equalities

$$x_i = \rho \xi_i, \quad (i = 1, 2, \ldots, n)$$

ρ being an arbitrary positive parameter.

This results in that each edge of the domain R is well determined by any point belonging to it.

Let us suppose that the domain R possesses s edges characterised by the points

$$(\xi_k) = (\xi_{1k}, \xi_{2k}, \ldots, \xi_{mk}). \quad (k = 1, 2, \ldots, s)$$

By declaring

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik}, \ \ (i = 1, 2, \ldots, m) \tag{3}$$

where

$$\rho_k \geq 0, \quad (k = 1, 2, \ldots, s) \tag{4}$$

one obtains a point (x) belonging to the domain R, the positive or zero parameters $\rho_1, \rho_2, \ldots, \rho_s$ being arbitrary.

10

Fundamental theorem. *Let us suppose that the inequalities (1) which define the domain R satisfy the condition* (Ξ)*.*

The domain R will be of m dimensions and each point belonging to it will be determined by the equalities (3) with condition (4).

The theorem introduced is well known in the case $m = 2$ and $m = 3$.

We will demonstrate that by supposing that the theorem be true in the case of $m - 1$ variables, the theorem will again be true in the case of m variables.

Let us examine first the various inequalities of the system (1). It can be the case that many among them could be put under the form

$$y_h(x) = \sum_{k=1}^{s} \rho_k^{(h)} y_k(x) \quad \text{where} \quad \rho_k^{(h)} \geq 0.$$

$$(k = 1, 2, \ldots, s; \rho_h^{(h)} = 0)$$

One will call such inequalities dependent and one will exclude them from the system (1).

Let us suppose that the system (1) contained only independent inequalities.

Their number ρ, on the ground of the supposition (2) made, will not be less than m.

This posed, let us examine a set P_h of points belonging to the domain R and verifying an equation

$$y_h(x) = 0, \tag{5}$$

the indice h having a value $1, 2, \ldots, \sigma$.

One will call "face of the domain R" the domain P_h.

On the ground of the supposition made, the face P_h will be in $m - 1$ dimensions.

To demonstrate this, let us make correspond to any point (x) verifying the equation (5) a point (u) in $m - 1$ coordinates $(u_1, u_2, \ldots, u_{m-1})$ by declaring

$$x_i = \sum_{j=1}^{m-1} \alpha_{ij} u_j. \ (i = 1, 2, \ldots, m)$$

The system of inequalities (1) will be transformed into a system

$$\eta_k(u) \geq 0 \ (k = 1, 2, \ldots, \sigma; k \neq h) \tag{7}$$

of inequalities in $m - 1$ variables $u_1, u_2, \ldots, u_{m-1}$.

Let us suppose that one knew how to reduce the equation

$$\sum_{k=1}^{\sigma} \rho_k \eta_k(u) = 0 \text{ where } \rho_h = 0 \text{ and } \rho_k \geq 0 \ (k = 1, 2, \ldots, \sigma) \tag{8}$$

into an identity. By virtue of (6), one will obtain the identity

$$\sum_{k=1}^{\sigma} \rho_k y_k(x) = \rho y_h(x) \quad \text{where} \quad \rho_h = 0.$$

One can not suppose that $\rho > 0$, since otherwise the inequality

$$y_h(x) \geq 0$$

would be dependent and on the ground of the supposition made would not belong to the system (1).

By supposing that $\rho \leq 0$, one will admit $\rho_h = -\rho$ and one will obtain the identity

$$\sum_{k=1}^{\sigma} \rho_k y_k(x) = 0 \text{ where } \rho_k \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

which is contrary to the hypothesis.

We have supposed that the theorem introduced be true in the case of $m - 1$ variables. As the equation (8) can not be reduced into an identity, one concludes that the system of inequalities (7) defines a domain $\mathcal{B}_h$ in $m-1$ dimensions. Moreover, by indicating with

$$(u_{11}, u_{21}, \ldots, u_{m-1,1}), (u_{12}, u_{22}, \ldots, u_{m-1,2}), \ldots, (u_{1t}, u_{2t}, \ldots, u_{m-1,t}) \tag{9}$$

the points which characterised t edges of the domain $\mathcal{B}_h$, one will determine any point (u) of this domain by the equalities

$$u_i = \sum_{k=1}^{t} \rho_k u_{ik} \text{ where } \rho_k \geq 0, \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m-1) \tag{10}$$

One will make correspond to the points (9) the points

$$(\xi_r) = (\xi_{1r}, \xi_{2r}, \ldots, \xi_{mr}), \quad (r = 1, 2, \ldots, t) \tag{11}$$

by determining them with the help of equalities (6) and (9).

The points obtained (11) characterise t edges of the domain R belonging to the face P_h. Any point (x) belonging to the face P_h will be determined, on the grounds of (6) and (10), by the equalities

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0. \ (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m) \tag{12}$$

Let us notice that all the points (11) verify the equation

$$y_h(x) = 0 \tag{13}$$

and satisfy the conditions

$$y_k(x) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

One obtains thus the equalities

$$y_h(\xi_r) \geq 0 \quad (r = 1, 2, \ldots, t; k = 1, 2, \ldots, \sigma) \tag{14}$$

The face P_h being in $m-1$ dimensions, the equalities (14) would define the coefficients of the equation (13) to a close by common factor.

11

Let us suppose that one had determined this way all the faces

$$P_1, P_2, \ldots, P_\sigma \tag{15}$$

in $m-1$ dimensions of the domain R.

Let us suppose that the points

$$(\xi_k) = (\xi_{1k}, \xi_{2k}, \ldots, \xi_{mk}) \ (k = 1, 2, \ldots, s) \tag{16}$$

characterise the various edges of the domain R belonging to the various faces (15).

By indicating

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0, \ (k = 1, 2, \dots, s; i = 1, 2, \dots, m) \tag{17}$$

one obtains a set of points which all belong to the domain R.

I say that any point (x) belonging to the domain R can be determined with the help of equalities (17).

One can suppose that the point (x) does not belong to any one of the faces (15), since any point belonging to them can be determined with the help of equalities (12).

By supposing that one had the inequalities

$$y_k(x) > 0, \quad (k = 1, 2, \dots, \sigma)$$

let us arbitrarily choose a point (ξ_r) among those of the series (16) and let us admit

$$x'_i = x_i - \rho \xi_{ir} \text{ where } \rho > 0. \ (i = 1, 2, \dots, m) \tag{18}$$

So long as the parameter ρ is sufficiently small, one will also have

$$y_k(x') > 0. \quad (k = 1, 2, \dots, \sigma)$$

By making the parameter increase in a continuous manner, one will determine with the help of equalities (18) a point (x') verifying an equation

$$y_h(x') = 0$$

and satisfying the condition

$$y_k(x') \geq 0. \quad (k = 1, 2, \dots, \sigma)$$

The point obtained (x') belongs to the face P_h, therefore one can declare

$$x'_i = \sum_{k=1}^{t} \rho'_k \xi_{ik} \text{ where } \rho'_k \geq 0. \ (k = 1, 2, \dots, t; i = 1, 2, \dots, m)$$

By virtue of (18), it becomes

$$x_i = \rho\xi_{ir} + \sum_{k=1}^{t} \rho'_k \xi_{ik} \text{ where } \rho > 0,\ \rho'_k \geq 0.$$

$$(k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

It remains to demonstrate that the domain F is in m dimensions.

Let us notice that all the points determined by the equalities (17) with condition

$$\rho_k > 0 \quad (k = 1, 2, \ldots, s)$$

are interior to the domain R.

In effect, all the points (16) verify the inequalities

$$y_h(\xi_k) \geq 0.\ (k = 1, 2, \ldots, s; h = 1, 2, \ldots, \sigma) \tag{19}$$

By multiplying these inequalities by ρ_k, let us make the sum of inequalities obtained; one will have, because of (17),

$$y_h(x) = \sum_{k=1}^{s} \rho_k y_h(\xi_k) \geq 0. \quad (h = 1, 2, \ldots, \sigma)$$

By virtue of (19), one will have the inequality

$$y_h(x) > 0,\ (h = 1, 2, \ldots, s) \tag{20}$$

so long as the numbers $y_h(\xi_1), y_h(\xi_2), \ldots, y_h(\xi_s)$ do not cancel each other out.

One can not suppose that the equalities

$$y_h(\xi_k) = 0 \quad (k = 1, 2, \ldots, s)$$

holds, because otherwise all the equations

$$y_1(x) = 0, y_2(x) = 0, \ldots, y_\sigma(x) = 0$$

would be of proportional coefficients, which is contrary to the hypothesis; therefore one will have the inequalities (20), and it follows that the domain R is of m dimensions.

We have demonstrated that the condition (Ξ) is sufficient for the domain R to be of m dimensions. It is easy to demonstrate that this condition is necessary.

12

We have defined in Number 10 the faces in $m-1$ dimensions of the domain R. This definition can be generalised.

Definition. *One will call face in μ dimensions of the domain R $(\mu = 1, 2, \ldots, m-1)$ a domain $P(\mu)$ formed from points belonging to the domain R and verifying a system of equations*

$$y_k(x) = 0, \quad (k = 1, 2, \ldots, \tau) \tag{21}$$

provided that these equations define a domain in μ dimensions composed of points which, all, do not verify any other equation $y_{\tau+1}(x) = 0, \ldots, y_\sigma(x) = 0$.

Let us choose among the points (16) all those which verify the equations (21).

By indicating with

$$\xi_k = (\xi_{1k}, \xi_{2k}, \ldots, \xi_{mk}), \qquad (k = 1, 2, \ldots, t)$$

one will declare

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0. \ (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m) \tag{22}$$

It is easy to demonstrate that any point (x) belonging to the face $P(\mu)$ can be determined with the help of equalities (22).

Corollary. *Each face of the domain R is a set of points determined by the equalities (22) provided that any point belonging to it could not be determined by the equalities*

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \quad \text{where} \quad \rho_k \geq 0,$$

$$(k = 1, 2, \ldots, s; i = 1, 2, \ldots, m)$$

unless all the parameters $\rho_{t+1}, \rho_{t+2}, \ldots, \rho_s$ do not cancel each other.

13

Any point belonging to the domain R either is interior to the domain R or is interior to a face of that domain.

Let us suppose that the point (x) be interior to a face $P(\mu)$ of the domain R which is formed from all the points determined by the equalities (22).

I argue that one can always determine the point (x) by the equalities (22) provided that

$$\rho_k > 0. \quad (k = 1, 2, \ldots, t)$$

To demonstrate this, let us indicate

$$\rho_i = \sum_{k=1}^{t} \xi_{ik}. \quad (i = 1, 2, \ldots, m)$$

The point (α) is interior to the face $P(\mu)$.

By admitting

$$x'_i = x_i - \rho\alpha_i \text{ where; } \rho > 0, \ (i = 1, 2, \ldots, m) \qquad (23)$$

one obtains a point (x'_i) which will be interior to the face $P(\mu)$ so long as the parameter ρ will be sufficiently small; it follows that

$$x'_i = \sum_{k=1}^{t} \rho'_k \xi_{ik} \quad \text{where} \quad \rho'_k \geq 0.$$

$$(k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

By virtue of (23), one obtains

$$x_i = \sum_{k=1}^{t} (\rho + \rho'_k)\xi_{ik}, \quad (i = 1, 2, \ldots, m)$$

and by making

$$\rho + \rho'_k = \rho_k, \quad (k = 1, 2, \ldots, t)$$

one will have

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \quad \text{where} \quad \rho_k > 0.$$

$$(k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

Let us notice that by making $\mu = m$ and $t = s$, one will indicate with the symbol $P(m)$ the domain R; one concludes that any point (x) which is interior to the domain R can be determined by the equalities

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \text{ where } \rho_k > 0.$$

$$(k = 1, 2, \ldots, s; i = 1, 2, \ldots, m)$$

On the correlative domains.

14

Definition. *Let us suppose that a domain R be determined with the help of inequalities*

$$p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

One will call correlative to the domain R the domain $\mathcal{R}$ which is formed from all the points (x) determined by the equalities

$$x_i = \sum_{k=1}^{\sigma} \rho_k p_{ik} \text{ where } \rho_k \geq 0. \ (k = 1, 2, \ldots, \sigma; i = 1, 2, \ldots, m) \tag{1}$$

I say that the domain $\mathcal{R}$ will be in m dimensions, if the domain R does not possess points verifying the equations

$$p_{1k}x_1 + p_{1k}x_2 + p_{mk}x_m = 0, \quad (k = 1, 2, \ldots, \sigma)$$

the point $x_1 = 0, x_2 = 0, \ldots, x_m = 0$ being excluded.

In effect, if all the points of the domain $\mathcal{R}$ verified the same equation

$$\xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_m x_m = 0,$$

one would have the equalities

$$\xi_1 p_{1k} + \xi_2 p_{2k} + \ldots + \xi_m p_{mk} = 0, \quad (k = 1, 2, \ldots, \sigma)$$

by virtue of (1), which is contrary to the hypothesis.

Theorem. *By supposing that the domain R be formed from all the points (x) determined by the equalities*

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0, \ (k = 1, 2, \ldots, s; i = 1, 2, \ldots, m) \tag{2}$$

one will define the correlative domain $\mathcal{R}$ with the help of inequalities.

$$\xi_{1k}x_1 + \xi_{2k}x_2 + \ldots + \xi_{mk}x_m \geq 0. \ (k = 1, 2, \ldots, s) \tag{3}$$

Let us indicate by $\mathcal{R}'$ the domain determined with the help of inequalities (3).

On the ground of the supposition made, all the points

$$(\xi_{11}, \xi_{21}, \ldots, \xi_{m1}), (\xi_{12}, \xi_{22}, \ldots, \xi_{m2}), \ldots, (\xi_{1s}, \xi_{2s}, \ldots, \xi_{ms})$$

characterise the edges of the domain R, and one will have the inequalities

$$p_{1h}\xi_{1k} + p_{2h}\xi_{2k} + \ldots + p_{mh}\xi_{mk}$$

$$(h = 1, 2, \ldots, \sigma; h = 1, 2, \ldots, \sigma). \qquad (4)$$

We have seen in Number 10 that each face P_h in $m-1$ dimensions of the domain R is characterised by the points

$$(\xi_{11}, \xi_{21}, \ldots, \xi_{m1}), (\xi_{12}, \xi_{22}, \ldots, \xi_{m2}), \ldots, (\xi_{1t}, \xi_{2t}, \ldots, \xi_{mt})$$

which verify the equation

$$y_h^x = 0 \qquad (5)$$

of the face P_h. One obtains the equalities

$$p_{1h}\xi_{1k} + p_{2h}\xi_{2k} + \ldots + p_{mh}\xi_{mk} = 0 \quad (k = 1, 2, \ldots, t)$$

which define the coefficients $p_{1h}, p_{2h}, \ldots, p_{mh}$ of the equation (5) to an immediate common factor.

One concludes, by virtue of the definition established in Number 9, that the point $(p_{1h}, p_{2h}, \ldots, p_{mh})$ characterises an edge of the domain $\mathcal{R}'$.

By attributing with the indice h the values $1, 2, \ldots, \sigma$, one obtains a series

$$(p_{11}, p_{21}, \ldots, p_{m1}), (p_{12}, p_{22}, \ldots, p_{m2}), \ldots, (p_{1\sigma}, p_{2\sigma}, \ldots, p_{m\sigma})$$

of points which characterise different edges of the domain $\mathcal{R}'$.

I argue that the domain $\mathcal{R}'$ does not possess other edges. To demonstrate this, let us suppose that a point $p_1, p_2, \ldots, p_m$ characterises an edge of the domain $\mathcal{R}'$.

One will have the equalities

$$p_1\xi_{1h} + p_2\xi_{2h} + \ldots + p_m\xi_{mh} = 0, \; (h = 1, 2, \ldots, t) \qquad (6)$$

which define the coefficients $p_1, p_2, \ldots, p_m$ to a nearby common factor, and one will have the inequalities

$$p_1\xi_{1h} + p_2\xi_{2h} + \ldots + p_m\xi_{mh} \geq 0. \; (k = 1, 2, \ldots, s) \qquad (7)$$

Let (x) be any point of the domain R. One will determine the point (x) with the help of equalities (2). By multiplying the inequalities (7) with ρ_k and by making the sum of inequalities obtained, one will have, because of (2),

$$p_1x_1 + p_2x_2 + \ldots + p_mx_m \geq 0.$$

One concludes that the inequalities

$$-p_1x_1 - p_2x_2 - \ldots - p_mx_m \geq 0$$

and

$$p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m \geq 0,$$

define a domain which is not in m dimensions.

By virtue of the fundamental theorem of Number 10, one will determine in this case positive values or zeros of parameters $\rho, \rho_1, \ldots, \rho_\sigma$ which reduce the equation

$$-\rho(p_1x_1 + p_2x_2 + \ldots + p_mx_m) + \sum_{k=1}^{\sigma} \rho_k(p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m) = 0$$

into an identity.

It follows that

$$p_i = \sum_{k=1}^{\sigma} \frac{\rho_k}{\rho} p_{ik} \text{ where } \frac{\rho_k}{\rho} \geq 0.44$$

$$(k = 1, 2, \ldots, \sigma; i = 1, 2, \ldots, m)$$

By substituting (6), one will have

$$\sum_{k=1}^{\sigma} \frac{\rho_k}{\rho}(\xi_{1h}p_{1k} + \xi_{2h}p_{2k} + \ldots + \xi_{mh}p_{mk}) = 0. \quad (h = 1, 2, \ldots, t)$$

By virtue of (4), one finds

$$\frac{\rho_k}{\rho}(\xi_{1h}p_{1k} + \xi_{2h}p_{2k} + \ldots + \xi_{mh}p_{mk}) = 0.$$

$$(h = 1, 2, \ldots, t; k = 1, 2, \ldots, \sigma)$$

Let us suppose that $\frac{\rho_k}{\rho} > 0$, then

$$\xi_{1h}p_{1k} + \xi_{2h}p_{2k} + \ldots + \xi_{mh}p_{mk} = 0, \quad (h = 1, 2, \ldots, t)$$

therefore the coefficients $p_1, p_2, \ldots, p_m$, by virtue of (6), are proportional to the coefficients $p_{1k}, p_{2k}, \ldots, p_{mk}$; it follows that the points $(p_{1k}, p_{2k}, \ldots, p_{mk})$ and $(p_1, p_2, \ldots, p_m)$ characterise the same edge of the domain $\mathcal{R}'$.

By virtue of the fundamental theorem in Number 10, all the points of the domain $\mathcal{R}'$ will be determined by the equality (1), this results in that the domains $\mathcal{R}$ and $\mathcal{R}'$ coincide.

Corollary. *Let us suppose that a face $P(\mu)$ in μ dimensions of the domain R be determined by the equations*

$$p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m = 0, \quad (k = 1, 2, \ldots, \tau)$$

and that any point (x) belonging to the face $P(\mu)$ be determined by the equalities

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0.$$

$$(k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

The correlative domain $\mathcal{R}$ will possess a corresponding face $\mathcal{B}(m - \mu)$ in $m - \mu$ dimensions determined by the equations

$$\xi_{1k}x_1 + \xi_{2k}x_2 + \ldots + \xi_{mk}x_m = 0 \quad (k = 1, 2, \ldots, t)$$

and any point (x) belonging to the face $\mathcal{B}(m - \mu)$ will be determined by the equalities

$$x_i = \sum_{k=1}^{\tau} \rho_k p_{ik} \text{ where } \rho_k \geq 0.$$

$$(k = 1, 2, \ldots, \tau; i = 1, 2, \ldots, m)$$

Definition of domains of quadratic forms corresponding to the various perfect forms

15

Let us consider any one perfect quadratic form φ.

Let us suppose that all the representations of the minimum of the form φ make up the series

$$(l_{11}, l_{21}, \dots, l_{n1}), (l_{12}, l_{22}, \dots, l_{n2}), \dots, (l_{1s}, l_{2s}, \dots, l_{ns}). \quad (1)$$

By indicating

$$\lambda_k = l_{1k}x_1 + l_{2k}x_2 + \ldots + l_{nk}x_n, \ (k = 1, 2, \dots, s) \qquad (2)$$

one corresponds to the series (1) a series of linear forms

$$\lambda_1, \lambda_2, \dots, \lambda_s.$$

Let us consider a domain R of quadratic forms determined by the equality

$$f(x_1, x_2, \dots, x_n) = \sum_{k=1}^{s} \rho_k \lambda_k^2$$

with condition that

$$\rho_k \geq 0. \quad (k = 1, 2, \dots, s)$$

One will say that the domain R correspond to the perfect form φ.

Let us notice that the domain R is in $\frac{n(n+1)}{2}$ dimensions.

By supposing the contrary let us suppose that all the quadratic forms belonging to the domain R verifies a linear equation

$$\Psi(f) = \sum p_{ij} a_{ij} = 0.$$

On the ground of the established definition, one will have the equalities

$$\Psi(\lambda_k^2) = 0 \quad (k = 1, 2, \ldots, s)$$

or, that which comes to the same thing, because of (2),

$$p_{ij} l_{ik} l_{jk} = 0 \quad (k = 1, 2, \ldots, s)$$

which is impossible, the form φ being perfect.

On the ground of what has been said in Number 9–14, the domain R possesses s edges characterised by the quadratic forms

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_s^2. \tag{3}$$

Let us suppose that one had determined all the faces

$$P_1, P_2, \ldots, P_\sigma$$

in $\frac{n(n+1)}{2} - 1$ dimensions of the domain R.

Each face P_k can be determined by two methods:

1. All the quadratic forms belonging to the face P_k verify an equation

$$\Psi_k(f) = \sum p_{ij}^{(k)} a_{ij} = 0$$

which can be determined in such a way that the inequality

$$\Psi_k(f) > 0$$

held so long as the form f belonging to the domain R is exterior to the face P_k.

2. By choosing among the quadratic forms (3) these

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_t^2$$

which verify the equation (4), one will determine all the quadratic forms belonging to the face P_k by the equalities

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{t} \rho_k \lambda_k^2,$$

where

$$\rho_k \geq 0. \quad (k = 1, 2, \ldots, t)$$

By virtue of the theorem of Number 14, the domain R can be considered as a set of points verifying the inequalities

$$\Psi_k(f) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

16 *On the extreme quadratic forms*

Let us indicate by $M(a_{ij})$ the minimum and by $D(a_{ij})$ the determinant of a positive quadratic form $\sum a_{ij}x_ix_j$. The positive quadratic form $\frac{1}{\sqrt[n]{D(a_{ij})}} \sum a_{ij}x_ix_j$ will be of determinant 1 and will possess the minimum

$$\frac{M(\alpha_{ij})}{\sqrt[n]{D(a_{ij})}} = \mathcal{M}(a_{ij}).$$

Let us examine the various value of the function

$$\mathcal{M}(a_{ij})$$

which is well determined in the set (f) of all the positive quadratic forms in n variables.

Definition. *One will call extreme ‡ a positive quadratic form $\sum a_{ij}x_ix_j$ which enjoys the property that the corresponding value of the function $\mathcal{M}(a_{ij})$ is minimum.*

Let us notice that the function $\mathcal{M}(a_{ij})$ does not change its value when one replaces quadratic form $\sum a_{ij}x_ix_j$ by a form of proportional coefficients.

By attributing to the coefficients of the extreme form $\sum a_{ij}x_ix_j$ variations

$$\epsilon_{ij} = \epsilon_{ji} \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

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‡ *See the Mémoire of Mr.'s Korkine and Zolotareff, Sur les formes quadratiques [On the quadratic forms], Mathematische Annalen V. VI, p. 368*

satisfying the condition

$$|\epsilon_{ij}| < \epsilon, \ \ (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \qquad (1)$$

ϵ being an arbitrary positive parameter, let us examine the corresponding value of the function $\mathcal{M}(a_{ij})$.

On the ground of the definition established, one can determine the parameter ϵ such that the inequality

$$\mathcal{M}(a_{ij} + \epsilon_{ij}) < \mathcal{M}(a_{ij}) \qquad (2)$$

held with condition (1) and so long as the coefficients ϵ_{ij} are not proportional to the coefficients

$$a_{ij}. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

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Theorem. *For a quadratic form* $\sum a_{ij}x_ix_j$ *to be extreme, it is necessary and sufficient that it be perfect and that its adjoint form* $\sum \frac{\partial D(a_{ij})}{\partial a_{ij}} x_ix_j$ *be interior to the domain corresponding to the form* $\sum a_{ij}x_ix_j$.

Let us indicate by

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}) \quad (3)$$

the various representations of the minimum $M(a_{ij})$ of the form $\sum a_{ij}x_ix_j$.

Let us consider a quadratic form $\sum(a_{ij} + \rho\epsilon_{ij})x_ix_j$, the parameter ρ being arbitrary. One can determine an interval

$$-\delta < \rho < \delta \text{ where } 0 < \delta < 1 \qquad (4)$$

such that all the representations of the minimum of the form $\sum(a_{ij} + \rho\epsilon_{ij})x_ix_j$ are found among the systems (3) so long as the variations ϵ_{ij} satisfy the condition (1).

By indicating with

$$M' = \sum(a_{ij} + \rho\epsilon_{ij})l_{ik}l_{jk} \text{ and } M = \sum a_{ij}l_{ik}l_{jk} \qquad (5)$$

the minima of forms $\sum(a_{ij}+\rho\epsilon_{ij})x_ix_j$ and $\sum a_{ij})x_ix_j$ and with D' and D their determinants, one will have

$$\mathcal{M}(a_{ij}+\rho\epsilon_{ij})=\frac{\sum(a_{ij}+\rho\epsilon_{ij})l_{ik}l_{jk}}{\sqrt[n]{D'}},\quad \mathcal{M}(a_{ij})=\frac{\sum a_{ij}l_{ik}l_{jk}}{\sqrt[n]{D}}.$$

By virtue of (2), one obtains the inequality

$$\frac{\sum(a_{ij}+\rho\epsilon_{ij})l_{ik}l_{jk}}{\sqrt[n]{D'}}<\frac{\sum a_{ij}l_{ik}l_{jk}}{\sqrt[n]{D}}$$

or, that which comes to the same thing,

$$\rho\sum\epsilon_{ij}l_{ik}l_{jk}<M\left(\sqrt[n]{\frac{D'}{D}}-1\right). \tag{6}$$

This declared, let us suppose that the form $\sum a_{ij}x_ix_j$ be not perfect.

One will determine in this case the variations ϵ_{ij} such that the equalities

$$\sum\epsilon_{ij}l_{ik}l_{jk}=0,$$

held. By virtue of (6), one will obtain the inequality

$$D'>D.$$

By developing the determinant D' into a series, one will have the inequality

$$\rho\sum\epsilon_{ij}\frac{\partial D}{\partial a_{ij}}+\frac{\rho^2}{2}\sum\epsilon_{ij}\epsilon_{kh}\frac{\partial^2 D}{\partial a_{ij}\partial a_{kh}}+\ldots>0. \tag{7}$$

The parameter ρ being arbitrary satisfying the condition (4), it is necessary tht

$$\sum\epsilon_{ij}\frac{\partial D}{\partial a_{ij}}=0.$$

Mr.'s Korkine and Zolotareff have demonstrated ‡ that in this case one will always have the inequality

$$\sum \epsilon_{ij}\epsilon_{kh}\frac{\partial^2 D}{\partial a_{ij}\partial a_{kh}} < 0,$$

therefore the inequality (7) is impossible.

18

We have demonstrated that the form $\varphi = \sum a_{ij}x_ix_j$ has to be perfect.

Let us suppose that the domain R corresponding to the perfect form φ be determined by σ inequalities

$$\Psi_r(f) = \sum p_{ij}^{(r)} a_{ij} \geq 0. \quad (r = 1, 2, \ldots, \sigma)$$

On the grounds of these inequalities, one will have

$$\Psi_r(\lambda_k^2) = \sum p_{ij}^{(r)} l_{ik} l_{jk} \geq 0. \ (k = 1, 2, \ldots, s; r = 1, 2, \ldots, \sigma) \tag{8}$$

Let us declare

$\epsilon_{ij} = tp_{ij}^{(r)}$ where $t > 0$. $\quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$

By virtue of (6), one will have

$$\rho t \sum p_{ij}^{(r)} l_{ik} l_{jk} < M\left(\sqrt[n]{\frac{D'}{D}} - 1\right). \tag{9}$$

Let us attribute to the parameter ρ a positive value satisfying the condition (4), by virtue of (8) and (9) there will arrive

$$D' > D.$$

By developing the determinant D' into a series, one obtains the inequality

$$\rho t \sum p_{ij}^{(r)} \frac{\partial D}{\partial a_{ij}} + \frac{(\rho t)^2}{2} \sum p_{ij}^{(r)} p_{ih}^{(r)} \frac{\partial^2 D}{\partial a_{ij}\partial a_{kh}} + \ldots > 0.$$

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‡ Mathematische Annalen, V. XI, p. 250

The positive parameter ρ being as small as one wish, it follows that

$$\sum p_{ij}^{(r)} \frac{\partial D}{\partial a_{ij}} > 0. \quad (r = 1, 2, \ldots, \sigma)$$

It is thus demonstrated that the form $\sum \frac{\partial D}{\partial a_{ij}} x_i x_j$, adjoint to the form φ, is interior to the domain R.

I argue that in this case the perfect form φ will be extreme.

By supposing the contrary, let us suppose that the inequality

$$\mathcal{M}(a_{ij} + \epsilon_{ij}) \geq \mathcal{M}(a_{ij}) \tag{10}$$

be verified by any one system of variations ϵ_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$ satisfying the condition (1) however small the parameter ϵ may be.

By virtue of (10), one obtains

$$\sum \epsilon_{ij} l_{ik} l_{jk} \geq M \left(\sqrt[n]{\frac{D'}{D}} - 1 \right); \quad (k = 1, 2, \ldots, s) \tag{11}$$

the inequality obtained has to hold whatever may te the value of the index $k = 1, 2, \ldots, s$.

By indicating

$$\eta_{ij} = a_{ij} \left(\sqrt[n]{\frac{D}{D'}} - 1 \right) + \epsilon_{ij} \sqrt[n]{\frac{D}{D'}},$$

$$(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{12}$$

let us examine the quadratic form

$$\varphi_0(x_1, x_2, \ldots, x_n) = \sum (a_{ij} + \eta_{ij}) x_i x_j. \tag{13}$$

By virtue of (12) the form φ_0 is of determinant D.

By choosing the parameter ϵ sufficiently small, one can suppose that

$$|\eta_{ij}| < \eta, (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{14}$$

η being a positive parameter as small as one would like.

By virtue of (5), (11) and (12), one obtains

$$\sum \eta_{ij} l_{ik} l_{jk} \geq 0. \; (k = 1, 2, \ldots, s) \tag{15}$$

By developing the determinant D of the form (13) into series, one will find

$$D + \sum \eta_{ij} \frac{\partial D}{\partial a_{ij}} + R_2 = D. \tag{16}$$

In this equality the remainder R_2 verifies an inequality

$$|R_2| < \eta^2 P,$$

P being a positive number not depending on the parameter η so long as $\eta < 1$.

By virtue of (16), one obtains

$$\left| \sum \eta_{ij} \frac{\partial D}{\partial a_{ij}} \right| < \eta^2 P. \tag{17}$$

We have suppose that the quadratic form $\sum \frac{\partial D}{\partial a_{ij}} x_i x_j$, adjoint to the form φ, be interior to the domain R. On the ground of that which has been said in Number 13, one will determine the form $\sum \frac{\partial D}{\partial a_{ij}} \cdot \, x_i x_j$ with the help of the equality

$$\sum \frac{\partial D}{\partial a_{ij}} x_i x_j = \sum_{k=1}^{s} \rho_k \lambda_k^2 \tag{18}$$

where

$$\rho_k > 0. \; (k = 1, 2, \ldots, s) \tag{19}$$

The equality (18) can be replaced by the following ones:

$$\frac{\partial D}{\partial a_{ij}} = \sum_{k=1}^{s} \rho_k l_{ik} l_{jk}. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By multiplying these equations by η_{ij} and by adding up the equalities obtained, one will have

$$\sum \eta_{ij} \frac{\partial D}{\partial a_{ij}} = \sum_{k=1}^{s} \rho_k \sum \eta_{ij} l_{ik} l_{jk}. \tag{20}$$

By virtue of (15), (17) and (19), one obtains the inequalities

$$0 \leq \sum \eta_{ij} l_{ik} l_{jk} < \eta^2 \frac{P}{\rho_k}; \quad (k = 1, 2, \ldots, s)$$

therefore one can admit

$$\sum \eta_{ij} l_{ik} l_{jk} = \tau_k \eta^2, \ (k = 1, 2, \ldots, s) \tag{21}$$

and the positive numbers or zeros τ_k $(k = 1, 2, \ldots, s)$ will not exceed fixed limits which do not depend on the parameter η.

After the definition of perfect forms, the equations (21) admit only a single system of solutions. By effecting this solution of equations (21), one obtains

$$\eta_{ij} = \tau_{ij} \eta^2 \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

where

$$|\tau_{ij}| < T, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

T being a positive number which does not depend on the parameter η; therefore one will have the inequalities

$$|\eta_{ij}| < \eta^2 T. \ (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{22}$$

This stated, let us take any one positive fraction ϑ and declare

$$\eta = \frac{\vartheta}{T}.$$

By virtue of (14), one will have

$$|\eta_{ij}| < \frac{\vartheta}{T}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

and because of (22), it will become

$$|\eta_{ij}| < \frac{\vartheta^2}{T}. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By admitting

$$\eta_{ij} < \frac{\vartheta^2}{T},$$

one will have, because of (22),

$$|\eta_{ij}| < \frac{\vartheta^4}{T}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

and so on.

One will obtain in this manner the inequalities

$$|\eta_{ij}| < \frac{\vartheta^{2k}}{T} \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; k = 0, 1, 2, \ldots)$$

it follows that

$$\eta_{ij} = 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By virtue of (12), one obtains

$$\epsilon_{ij} = a_{ij}\left(\sqrt[n]{\frac{D'}{D}} - 1\right); \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

therefore the coefficients ϵ_{ij} are proportional to the coefficients a_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$, which is contrary to the hypothesis.

Properties of the set of domains corresponding to the various perfect forms in n variables.

19

Any perfect form φ will be transformed by an equivalent perfect form φ' with the help of any substitution S of integer coefficients and of determinant ±1.

Let us indicate by R and R' the domains corresponding to the perfect forms φ and φ' and by T the substitution adjoint to the substitution S.

One will easily demonstrate that the domain R will be transformed into an equivalent domain R' with the help of the substitution T.

One concludes that there exists a finite number of domains equivalent to the domain R.

Let us indicate by (R) the set of all the domains corresponding to the various perfect forms in n variables.

The set (R) can be divided into classes of equivalent domains.

On the ground of that which has been previously said, the number of different classes of the set (R) is equal to the number of classes of perfect forms in n variables.

20

Theorem. *Let us suppose that a quadratic form f be interior to a face $P(\mu)$ in μ dimensions of the domain R $(\mu = 1, 2, \dots, \frac{n(n+1)}{2})$.*

The form f will belong only to the domains of the set (R) which are contiguous through the face $P(\mu)$.

Let us suppose that the domain R be characterised by the quadratic form

$$\lambda_1^2, \lambda_2^2, \dots, \lambda_s^2 \tag{1}$$

and that the face $P(\mu)$ in μ dimensions of the domain R be characterised by the quadratic forms

$$\lambda_1^2, \lambda_2^2, \dots, \lambda_t^2. \tag{2}$$

In the case $\mu = \frac{n(n+1)}{2}$, one will admit $t = s$, and the symbol $P\left(\frac{n(n+1)}{2}\right)$ will indicate the domain R.

The quadratic form f being interior to the face $P(\mu)$, one will have the equality

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{t} \rho_k \lambda_k^2 \text{ where } \rho_k > 0. \ (k = 1, 2, \ldots, t) \tag{3}$$

Let us suppose that the same form f belonged to another domain R' of the set (R).

Let us suppose that the domain R' be characterised by the quadratic forms

$$\lambda_1'^2, \lambda_2'^2, \ldots, \lambda_\sigma'^2 \tag{4}$$

and that the form f be interior to the face $P'(\nu)$ of the domain R' characterised by the quadratic forms

$$\lambda_1'^2, \lambda_2'^2, \ldots, \lambda_\tau'^2. \tag{5}$$

One will have, on the ground of the supposition made,

$$f(x_1, x_2, \ldots, x_n) = \sum_{h=1}^{\tau} \rho_h' \lambda_h'^2 \text{ where } \rho_h' > 0. \ (h = 1, 2, \ldots, \tau) \tag{6}$$

This declared, let us indicate by φ and φ' the perfect forms corresponding to the domains R and R' and suppose, for more simplicity, that the minimum of forms φ and φ' be M.

By indicating with the symbol (f, f') the result

$$(f, f') = \sum a_{ij} a_{ij}',$$

from two quadratic forms

$$f(x_1, x_2, \ldots, x_n) = \sum a_{ij} x_i x_j$$

and

$$f'(x_1, x_2, \ldots, x_n) = \sum a_{ij}' x_i x_j,$$

let us examine two results (f, φ) and (f, φ').

By virtue of (13), one obtains

$$(f,\varphi)=\sum_{k=1}^{t}\rho_k(\varphi,\lambda_k^2)\ \text{ and }\ (f,\varphi')=\sum_{k=1}^{t}\rho_k(\varphi',\lambda_k^2). \tag{7}$$

By virtue of (6), one obtains

$$(f,\varphi)=\sum_{h=1}^{\tau}\rho'_h(\varphi,{\lambda'_h}^2)\ \text{ and }\ (f,\varphi')=\sum_{h=1}^{\tau}\rho'_h(\varphi',{\lambda'_h}^2). \tag{8}$$

Let us notice that

$$(\varphi,\lambda_k^2)=M\ \text{ and }\ (\varphi',\lambda_k^2)\geq M;\ \ (k=1,2,\ldots,s) \tag{9}$$

$$(\varphi,{\lambda'_h}^2)\geq M\ \text{ and }\ (\varphi',{\lambda'_h}^2)=M.\ \ (h=1,2,\ldots,\sigma) \tag{10}$$

From equalities (7), one derives

$$(f,\varphi')-(f,\varphi)=\sum_{k=1}^{t}\rho_k\left[(\varphi',\lambda_k^2)-(\varphi,\lambda_k^2)\right], \tag{11}$$

and by virtue of (3) and (9) there comes

$$(f,\varphi')-(f,\varphi)\geq 0.$$

From equalities (8), one derives

$$(f,\varphi')-(f,\varphi)=\sum_{h=1}^{\tau}\rho'_h\left[(\varphi',{\lambda'_h}^2)-(\varphi,{\lambda'_h}^2)\right], \tag{12}$$

and by virtue of (6) and (10), one will have

$$(f,\varphi')-(f,\varphi)\leq 0.$$

It follows that

$$(f,\varphi')=(f,\varphi),$$

and the equalities (11) and (12) give

$$(\varphi', \lambda_k^2) = (\varphi, \lambda_k^2), \quad (k = 1, 2, \ldots, t)$$
$$(\varphi', {\lambda'_h}^2) = (\varphi, {\lambda'_h}^2). \quad (h = 1, 2, \ldots, \tau)$$

By virtue of (9) and (10), there arrive

$$(\varphi, {\lambda'_h}^2) = M, \quad (h = 1, 2, \ldots, \tau) \tag{13}$$
$$(\varphi', \lambda_k^2) = M. \quad (k = 1, 2, \ldots, t) \tag{14}$$

By virtue of equalities (13), the quadratic forms (5) are found among those of the series (1). By virtue of (14), the quadratic forms (2) are found among those of the series (4).

I argue that in this case the series (2) and (5) contain the same forms.

To demonstrate this, let us suppose that all the forms belonging to the face $P(\mu)$ verify the equations

$$\Psi_1(f) = 0,\ \Psi_2(f) = 0, \ldots,\ \Psi_r(f) = 0$$

and that any form belonging to the domain R verifies the inequalities

$$\Psi_1(f) \geq 0,\ \Psi_2(f) \geq 0, \ldots,\ \Psi_r(f) \geq 0. \tag{15}$$

By virtue of (6), one will have

$$\rho'_1 \Psi_i({\lambda'_1}^2) + \rho'_2 \Psi_i({\lambda'_2}^2) + \ldots + \rho'_\tau({\lambda'_\tau}^2) = 0, \quad (i = 1, 2, \ldots, r)$$

and because of (15), one finds

$$\Psi_i({\lambda'_h}^2) = 0; \quad (i == 1, 2, \ldots, r; h = 1, 2, \ldots, \tau)$$

therefore all the forms of the series (5) belong to the series (2).

In the same way, one will demonstrate that all the forms of the series (2) belong to the series (5).

One concludes that the faces $P(\mu)$ and $P'(\nu)$ coincide, therefore the domains R and R' are contiguous through the face $P(\mu)$.

Corollary. *A quadratic form which is interior to a domain of the set* (R) *can not belong to any other domain of that set.*

21

Theorem. *Let us suppose that to a face* $P(\mu)$ *of the domain* R *belong positive quadratic forms. In this case, the number of domains of the set* (R) *contiguous through the face* $P(\mu)$ *is finite.*

Let us indicate by

$$R, R_1, R_2, \ldots$$

the domains of the set (R) contiguous through the face $P(\mu)$. Let

$$\varphi, \varphi_1, \varphi_2, \cdots$$

be the corresponding perfect forms having the minimum M.

On the ground of the supposition made, one positive quadratic form f will be interior to the face $P(\mu)$.

We have demonstrated in the previous number that

$$(f, \varphi) = (f, \varphi_1) = (f, \varphi_2) = \ldots \tag{16}$$

It is easy to demonstrate that the number of perfect forms having the minimum M and verifying the equalities (16) is finite.

Algorithm for the search for the domain of the set (R) *contiguous to another domain by a face in* $\frac{n(n+1)}{2} - 1$ *dimensions*

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Let

$$\varphi(x_1, x_2, \ldots, x_n) = \sum a_{ij} x_i x_j$$

be a perfect form having the minimum M the various representation of which make up a series

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}). \quad (1)$$

Let us suppose that a face P of the domain R corresponding to the perfect form φ be determined by the equation

$$\Psi(f) = \sum p_{ij} a_{ij} = 0$$

and by the condition

$$\Psi(f) \geq 0$$

which is verified by any quadratic form belonging to the domain R.

Let us suppose that the face P be characterised by the quadratic forms

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_t^2 \quad (2)$$

where

$$\lambda_k = l_{1k} x_1 + l_{2k} x_2 + \ldots + l_{nk} x_n. \quad (k = 1, 2, \ldots, s)$$

On the ground of the supposition made, one will have the equalities

$$\sum p_{ij} l_{ik} l_{jk} = 0 \;\; (k = 1, 2, \ldots, t) \quad (3)$$

which define the coefficients P_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$ to an immediate common factor.

Let us suppose that the face P could belong to the other domains of the set (R). Let us indicate by R' a similar domain. Let φ' be the perfect form corresponding to the domain R'.

By virtue of the supposition made, the quadratic form (2) belong to the domains R and R'. It results in that the systems

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, [\ldots,] l_{n2}), \ldots, (l_{1t}, l_{2t}, \ldots, l_{nt}) \quad (4)$$

corresponding to the forms (2) represent the minimum of forms φ and φ'.

Let us suppose, for more simplicity, that the forms φ and φ' had the minimum M. One will have the equalities

$$\sum a_{ij}l_{ik}l_{jk} = M \text{ and } \sum a'_{ij}l_{ik}l_{jk} = M, \quad (k = 1, 2, \ldots, t) \quad (5)$$

by putting

$$\varphi'(x_1, x_2, \ldots, x_n) = \sum a'_{ij}x_ix_j.$$

From equation (5), one gets

$$\sum (a'_{ij} - a_{ij})l_{ik}l_{jk} = 0, \quad (k = 1, 2, \ldots, t)$$

and by virtue of (3), it becomes

$$a'_{ij} = a_{ij} + \rho p_{ij}. \ (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \qquad (6)$$

Let us indicate

$$\Psi(x_1, x_2, \ldots, x_n) = \sum p_{ij}x_ix_j.$$

By virtue of (16), one obtains

$$\varphi'(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n). \qquad (7)$$

This stated, let us choose in the series (1) a system $(l_{1h}, l_{2h}, \ldots, l_{nh})$ which does not belong to the series (4). As

$$\varphi(l_{1h}, l_{2h}, \ldots, l_{nh}) = M \quad \varphi'(l_{1h}, l_{2h}, \ldots, l_{nh}) \geq M$$

and

$$\Psi(l_{1h}, l_{2h}, \ldots, l_{nh}) > 0,$$

one deduces from the equality

$$\varphi'(l_{1h}, l_{2h}, \ldots, l_{nh}) = \varphi(l_{1h}, l_{2h}, \ldots, l_{nh}) + \rho\Psi(l_{1h}, l_{2h}, \ldots, l_{nh})$$

the inequality

$$\rho \geq 0.$$

The supposition $\rho = 0$ being obviously impossible, one obtains

$$\rho > 0,$$

and it follows that

$$\varphi'(l_{1h}, l_{2h}, \ldots, l_{nh}) > M.$$

Let us indicate by

$$(l'_1, l'_2, \ldots, l'_n), (l''_1, l''_2, \ldots, l''_n), \ldots, (l_1^{(r)}, l_2^{(r)}, \ldots, l_n^{(r)}) \qquad (8)$$

all the representations of the minimum of the perfect form φ' which are not found in the series (4). By virtue of (7), one will have

$$\varphi'(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) =$$

$$\varphi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) + \rho\Psi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)})$$

$$(k = 1, 2, \ldots, r)$$

which results in

$$\varphi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) > M \text{ and } \Psi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) < 0.$$

$$(k = 1, 2, \ldots, r) \qquad (9)$$

The value of the parameter ρ will have for expression

$$\rho = \frac{\varphi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) - 1}{-\Psi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)})}. \quad (k = 1, 2, \ldots, r)$$

Let us examine any one value of the function

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{\Psi(x_1, x_2, \ldots, x_n)} \qquad (10)$$

determined with the condition

$$\Psi(x_1, x_2, \ldots, x_n) < 0. \qquad (11)$$

I argue that one will have the inequality

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{\Psi(x_1, x_2, \ldots, x_n)} \geq \rho.$$

Let us suppose the contrary. By supposing that

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{\Psi(x_1, x_2, \ldots, x_n)} < \rho,$$

one will find, because of (11),

$$\varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n) < M$$

or, that which comes to the same thing because of (7),

$$\varphi'(x_1, x_2, \ldots, x_n) < M,$$

which is contrary to the hypothesis.

We have arrived at the following important result:

There exists only a single domain R' contiguous to the domain R through the face P. The corresponding perfect form φ' will be determined by the equality (7) provided that the parameter ρ presents the smallest positive value of the function (10).

Let us notice that by virtue of (3) and (9), all the quadratic forms belonging to the domain R' verify the inequality

$$\Psi(f) \leq 0.$$

One concludes that the domains R and R' are found from two opposite sides of the plane in $\frac{n(n+1)}{2} - 1$ dimensions determined by the equation

$$\Psi(f) = 0.$$

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The smallest positive value of the function (10) can be obtained with the help of operations the number of which is finite.

The whole problem is reduced to the preliminary study of a system $(l_1, l_2, \ldots, l_n)$ of integers verifying the inequality

$$\Psi(l_1, l_2, \ldots, l_n) < 0$$

and satisfying the condition that the quadratic form

$$\varphi_0(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho_0 \Psi(x_1, x_2, \ldots, x_n),$$

where one has admitted

$$\rho_0 \frac{\varphi(l_1, l_2, \ldots, l_n) - M}{-\Psi(l_1, l_2, \ldots, l_n)}$$

be positive.

One will determine in this case all the systems $(x_1, x_2, \ldots, x_n)$ of integers verifying the inequality

$$\varphi(x_1, x_2, \ldots, x_n) \geq M \tag{12}$$

the number of which is finite, and one will find among these systems all those which define the smallest value of the function (10).

Let us indicate by R, as we have done in Number 2, the upper limit of values of the parameter ρ.

The problem is reduced to the study of a system $(l_1, l_2, \ldots, l_n)$ of integers verifying the inequality

$$\varphi(l_1, l_2, \ldots, l_n) + R\Psi(l_1, l_2, \ldots, l_n) < M. \tag{13}$$

It can turn out that the equation

$$\varphi(x_1, x_2, \ldots, x_n) + R\Psi(x_1, x_2, \ldots, x_n) = 0$$

will be verified by integers, one will determine them with the help of equations

$$\frac{\partial \varphi}{\partial x_i} + R\frac{\partial \Psi}{\partial x_i} = 0. \quad (i = 1, 2, \ldots, n)$$

In the case where these equations can not be verified by any one system of integers, one will study the values of linear forms

$$\frac{\partial \varphi}{\partial x_i} + R\frac{\partial \Psi}{\partial x_i} \quad (i = 1, 2, \ldots, n)$$

and one will determine as many as one wish of the systems of integers verifying the inequality (13).

By supposing that a system of integers $(l_1, l_2, \ldots, l_n)$ verifying the inequality (13) were determined, one can look for the smallest positive value ρ of the function (10) with the help of the following procedure.

The inequality (12) can be put under the form

$$\varphi(x_1, x_2, \ldots, x_n)\left(1 - \frac{\rho_0}{R}\right) + \frac{\rho_0}{R}\left[\varphi(x_1, x_2, \ldots, x_n) + \right.$$
$$\left. R\Psi(x_1, x_2, \ldots, x_n)\right] \leq M,$$

and as

$$\varphi(x_1, x_2, \ldots, x_n) + R\Psi(x_1, x_2, \ldots, x_n) \geq 0,$$

it becomes

$$\varphi(x_1, x_2, \ldots, x_n)\left(1 - \frac{\rho_0}{R}\right) \leq M,$$

or differently

$$\varphi(x_1, x_2, \ldots, x_n) \leq M \frac{R}{R - \rho_0}.$$

Among the systems of integers verifying this inequality one will find all the systems (8) searched for.

Algorithm for the search for the domain of the set (R) to which belongs an arbitrary positive quadratic form. 24

Theorem. Any positive quadratic form belongs to at least one domain of the set (R).

Let

$$f(x_1, x_2, \ldots, x_n) = \sum a_{ij} x_i x_j$$

be any one positive quadratic form.

Let us choose any one domain R from the set (R).

Let us suppose that the form f, did not belong to the domain (R).

In that case all the linear inequalities which defined the domain R will not be verified. Let us suppose that one had the inequality

$$\Psi(f) = \sum p_{ij} a_{ij} < 0. \tag{1}$$

Let us indicate by R_1 the domain contiguous to the domain R through the face in $\frac{n(n+1)}{2} - 1$ dimensions determined by the equation

$$\Psi(f) = 0.$$

By indicating with φ and φ_1 the contiguous perfect forms corresponding to the domains R and R_1, one will have, as we have seen in Number 22,

$$\varphi_1(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n) \tag{2}$$

where $\rho > 0$ and $\Psi(x_1, x_2, \ldots, x_n) = \sum p_{ij} x_i x_j$.

Let us examine two results (f, φ) and (f, φ_1). By virtue of (2), one will have

$$(f, \varphi_1) = (f, \varphi) + \rho(f, \Psi),$$

and as, because of (1),

$$(f, \Psi) = \sum p_{ij} a_{ij} < 0,$$

it becomes

$$(f, \varphi) > (f, \varphi_1).$$

Let us suppose that by proceeding in this manner one obtains a series of domains

$$R, R_1, R_2, \ldots. \tag{3}$$

By indicating with

$$\varphi, \varphi_1, \varphi_2, \ldots \tag{4}$$

the series of corresponding perfect forms, one will have the inequalities

$$(f,\varphi) > (f,\varphi_1) > (f,\varphi_2) > \cdots$$

so long as the form f did not belong to the domains (3).

By noticing that all the perfect forms (4) possess the same minimum M, one will easily demonstrate that the number of perfect forms (4) verifying the integrality

$$(f,\varphi) < P$$

is bounded, whatever may be the value of the positive parameter P.

One concludes that the series of domains (3) will necessarily be terminated by a domain R_m to which belonged the form f considered.

Study of a complete system of domains representing the various classes of the set (R).

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Let R be any one domain of the set (R). Let us suppose that one had determined all the domains

$$R, R_1, R_2, \ldots, R_\sigma \tag{1}$$

contiguous to the domain R through the various faces in $\frac{n(n+1)}{2} - 1$ dimensions, then let us suppose that one had determined all the domains contiguous to the domains (1) and so on.

I say that by proceeding in this manner one will come across any domain of the set (R) arbitrarily chosen.

For example, if one wish to arrive at a domain $R^{(0)}$, one will choose a positive quadratic form f which is interior to the domain $R^{(0)}$ and one will proceed as we have done in Number 24. One will determine this way a series of domains

$$R, R', R'', \ldots, R^{(\mu)}, R^{(0)}$$

which are successively contiguous through faces in $\frac{n(n+1)}{2}$ -1 dimensions.

We have seen in Number 19 that the set (R) can be divided into classes of equivalent domains the number of which is finite.

Let us find a system of domains representing the various classes of the set (R).

By starting from the domain R, we have determined all the domains

$$R_1, R_2, \ldots, R_\sigma$$

contiguous to the domain R. By not considering equivalent domains as being different, let us choose among the domain (1) those which are not one to one equivalent and are not equivalent to the domain R. Let us suppose that one had obtained the series

$$R, R_1, R_2, \ldots, R_{\mu-1} \tag{2}$$

of domains which are not one to one equivalent.

One will study in the same way the domains contiguous to the domains $R, R_1, R_2, \ldots, R_{\mu-1}$ and one will extend the series (2) by adding to it new domains

$$R_\mu, R_{\mu+1}, \ldots, R_{\nu-1}$$

which are not one to one equivalent and are not equivalent to the domains (2).

By proceeding in this way, one will always obtain a series

$$R, R_1, R_2, \ldots, R_{\tau-1} \tag{3}$$

which enjoys the following property: the domain belonging to the series (3) are not one to one equivalent and all the domains contiguous to the domains (3) are equivalent to them.

The series (3) obtained presents a complete system of representations of various classes of the set (R).

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The study of the series (3) can be facilitated particularly by the help of substitutions which transform into itself the domains of the set (R).

Let us suppose that the domain R corresponding to a perfect form φ be determined by the inequalities

$$\sum p_{ij}^{(k)} a_{ij} \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

By declaring

$$\Psi_k(x_1, x_2, \ldots, x_n) = \sum p_{ij}^{(k)} x_i x_j, \quad (k = 1, 2, \ldots, \sigma)$$

one will determine, as we have seen in Number 22, by the equalities

$$\varphi_k = \varphi + \rho_k \Psi_k \quad (k = 1, 2, \ldots, \sigma) \tag{4}$$

σ perfect forms $\varphi_1, \varphi_2, \ldots, \varphi_\sigma$. One will call them contiguous to the perfect form φ.

Let us indicate by g the group of substitutions which do not change the perfect form φ.

The perfect forms $\varphi_1, \varphi_2, \ldots, \varphi_\sigma$ being well determined by the perfect form φ, one concludes that all the substitutions of the group g will only permute the forms $\varphi_1, \varphi_2, \ldots, \varphi_\sigma$.

By not considering as different the forms in proportional coefficients, one can say, by virtue of (4), that the group g will only permute the quadratic forms

$$\Psi_1, \Psi_2, \ldots, \Psi_\sigma. \tag{5}$$

Let us suppose that one had chosen in this series the forms

$$\Psi_1, \Psi_2, \ldots, \Psi_{\mu-1} \tag{6}$$

which enjoyed the following properties: each form of the series (5) will be transformed into a form of the series (6) with the help of a substitution belonging to the group

g, the forms (6) can not be transformed one to one with the help of substitutions of the group g.

The perfect forms

$$\varphi_k = \varphi + \rho_k \Psi_k \quad (k = 1, 2, \ldots, \mu - 1)$$

can replace the perfect forms (4), therefore one will determine only the values of parameters $\rho_1, \rho_2, \ldots, \rho_{\mu-1}$.

The corresponding domains

$$R_1, R_2, \ldots, R_{\mu-1}$$

can replace the domains (1).

It can come to pass that among the domains $R, R_1, R_2, \ldots, R_{\mu-1}$ are found equivalent domains, one will recognise this with the help of particular methods.

27 *On a reduction method of positive quadratic forms.*

Definition. *One will call reduced any positive quadratic form belonging to any one domain*

$$R, R_1, R_2, \ldots, R_{\tau-1} \tag{1}$$

of a complete system of representations of various classes of the set (R).

Let us suppose that one had determined all the substitutions

$$S_1, S_2, \ldots, S_m \tag{2}$$

which transform the domains contiguous with the domains (1) through the faces in $\frac{n(n+1)}{2} - 1$ dimensions into these domains here.

Let f be any one positive quadratic form which is not reduced. One will determine with the help of the algorithm shown in Number 24 a series of domains

$$R, R', R'', \ldots, R^{(h)}$$

successively contiguous. Let us suppose that the domain $R^{(h)}$ be the first one which does not belong to the series (1).

With the help of a substitution S' which is found among those of the series (2), one will transform the domain $R^{(h)}$ into a domain R_k belonging to the series (1).

By transforming the form f with the help of the substitution S' into an equivalent form f', one will determine with the help of the form f' a new series of domains

$$R_k, R'_k, \ldots, R_k^{(t)}$$

and so on.

One will determine in this way a series of substitutions

$$S', S'', \ldots, S^{(\lambda)}$$

which, all, are found among those of the series (2) and the product

$$S = S'S'' \cdots S^{(\lambda)}$$

of which presents a substitution S with the help of which the form f will be transformed into a reduced form.

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Let us suppose now that two reduced forms f and f' be equivalent.

If one of these forms, for example f, is interior to the domain R_k, the form f' will also be interior to it. One concludes that the form f can be transformed into a form f' only with the help of a substitution which transforms the domain R_k into itself.

Let us suppose that the reduced equivalent forms f and f' be interior to the faces in μ dimensions of domains (1).

In this case one will declare supplementary conditions for the reduced forms f and f'. After having determined all the faces in μ dimensions of domains (1), one will choose a complete system of representatives of these various classes. Let us suppose that this system be formed by the faces in μ dimensions

$$P_1(\mu), P_2(\mu), \ldots, P_\nu(\mu). \qquad (3)$$

Any positive quadratic form which is interior to a face in μ dimensions of a domain of the set (R) will be equivalent to a form which is interior to the faces (3), one will call it reduced.

Two reduced positive quadratic forms which are interior to the faces (3) will be equivalent only provided that they be interior to the same face and that the substitution which transforms one of them into another one also transforms this face into itself.

We have arrived at the following result:

A reduced quadratic form can be transformed into another reduced form or into itself only with the help of a substitution which transforms into itself a domain or a face of domains belonging to the series (1).

Second Part

Some applications of the general theory to the study of perfect quadratic forms

29

On the principal perfect form

We will not consider as different the quadratic forms of proportional coefficients, therefore one can arbitrarily choose the minimum value of a positive quadratic form.

In that which follows, one will study only the perfect quadratic forms whose minimum is 1. One will indicate by D the determinant of these forms.

Among the various perfect forms, one form

$$\varphi = x_1^2 + x_2^2 + \ldots + x_n^2 + x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n \; ^{\ddagger}$$

where

$$a_{ii} = 1,\ (i = 1, 2, \ldots, n),$$

$$a_{ij} = \frac{1}{2},\ (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

and

$$D = \frac{n+1}{2^n}$$

One will call principal the perfect form φ.

The perfect form φ possesses $\frac{n(n+1)}{2}$ representations of the minimum 1, which define $\frac{n(n+1)}{2}$ linear forms

$$\lambda_1 = x_1,\ \lambda_2 = x_2, \ldots,$$

$$\lambda_n = x_n,\ \lambda_{n+1} = x_1 - x_2,\ \lambda_{n+2} = x_1 - x_3, \ldots,$$

$$\lambda_{\frac{n(n+1)}{2}} = x_{n-1} - x_n.$$

The domain R corresponding to the perfect form φ is made up of all the quadratic forms determined by the equality

$$\sum a_{ij}x_ix_j = \sum_{k=1}^{\frac{n(n+1)}{2}} \rho_k\lambda_k^2 \text{ where } \rho_k \geq 0.$$

$$(k = 1, 2, \ldots, \frac{n(n+1)}{2})$$

From this equality one obtains

$$\rho_k = a_{1k} + a_{2k} + \ldots + a_{nk} \text{ so long as } k = 1, 2, \ldots, n,$$

$$\rho_k = -a_{ij} \text{ so long as } k > n;$$

The form φ has been given for the first time by Zolotareff in a Mémoire titled: On an indeterminate equation of the third degree (in Russian)

$$(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

therefore the domain R will be determined by the following inequalities:

$$\begin{cases} a_{1k} + a_{2k} + \ldots + a_{nk} \geq 0, \quad (k = 1, 2, \ldots, n) \\ -a_{ij} \geq 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j) \end{cases} \tag{1}$$

By virtue of (1), the perfect form φ possesses $\frac{n(n+1)}{2}$ contiguous perfect forms which are determined by the equalities

$$\begin{cases} \varphi_k = \varphi + \rho_k x_k (x_1, x_2, \ldots, x_n), \quad (k = 1, 2, \ldots, n) \\ \varphi_k = \varphi - \rho_k x_i x_j, \quad \left(k = n+1, n+2, \ldots, \dfrac{n(n+1)}{2}\right), \\ \qquad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j) \end{cases} \tag{2}$$

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Let us find equivalent forms among the perfect forms contiguous to the perfect form φ.

To this effect, let us determine the group g of substitutions which do not change the form φ.

Let us examine, in the first place, the form adjoint to the form φ.

One will easily demonstrate that the coefficients of the form adjoint to φ are proportional to those of the form

$$\omega = \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_{\frac{n(n+1)}{2}}^2. \tag{3}$$

One concludes, by virtue of the theorem of Number 17, that the principal perfect form φ is extreme.

The quadratic form ω will have for expression

$$\omega = nx_1^2 + nx_2^2 + \ldots + nx_n^2 - 2x_1x_2 - 2x_1x_3 - \ldots - 2x_{n-1}x_n$$

where

$$a_{ii} = n, \ (i = 1, 2, \ldots, n) \quad a_{ij} = -1.$$

$$(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

Let us find all the representations of the minimum of the form ω.

The linear forms

$$x_1, x_2, \ldots, x_n, x_1 + x_2 + \ldots + x_n \tag{4}$$

characterise $n + 1$ representations of the value n of the form ω.

I say that the form ω has the minimum n and all the representations of the minimum of the form ω are characterised by the linear form (4).

To demonstrate this, let us examine any one value $\omega(x_1, x_2, \ldots, x_n)$ of the form ω. By supposing that none of the numbers $x_1, x_2, \ldots, x_n$ becomes zero, one will have by virtue of (13)

$$\omega(x_1, x_2, \ldots, x_n) > n,$$

the system $x_1 = 1$, $x_2 = 1, \ldots, x_n = 1$ being excluded.

Let us suppose that any one of the numbers $x_1, x_2, \ldots,$ x_n does not cancel out and that

$$x_{k+1} = 0, \ x_{k+2} = 0, \ldots, \ x_n = 0;$$

one obtains, by virtue of (3),

$$\omega(x_1, x_2, \ldots, x_k, 0, \ldots, 0) =$$

$$(n - k + 1)(x_1^2 + x_2^2 + \ldots + x_k^2) + \sum(x_k - x_h)^2,$$

and it follows that

$$\omega(x_1, x_2, \ldots, x_n) \geq k(n - k + 1),$$

therefore

$$\omega(x_1, x_2, \ldots, x_n) > n \quad \text{so long as} \quad k \geq 2.$$

This stated, let us indicate by G the group of substitutions which transform into itself the domain R. By virtue of (3), any substitution of the group G does not change the form ω.

The group g being adjoint to the group G, one concludes that each substitution of the group g will only permute te linear forms (4) by changing the sign of a few among them.

By noticing that

$$x_1^2 + x_2^2 + \ldots + x_n^2 + (x_1 + x_2 + \ldots + x_n)^2 = 2\varphi,$$

one concludes that the group g is composed of all the substitutions which permute the forms

$$x_1^2 + x_2^2 + \ldots + x_n^2 + (x_1 + x_2 + \ldots + x_n)^2.$$

Let us indicate

$$x_0 = -x_1 - x_2 - \ldots - x_n \text{ and } x'_0 = -x'_1 - x'_2 - \ldots - x'_n, \quad (5)$$

and let $k_0, k_1, \ldots, k_n$ be any one permutation of numbers $0, 1, 2, \ldots, n$.

By posing

$$x_i = e_i x'_{k_i} \quad \text{where} \quad e_i = \pm 1, \ (i = 0, 1, 2, \ldots, n) \quad (6)$$

one will have

$$x_0 + x_1 + \ldots + x_n = e_0 x'_{k_0} + e_1 x'_{k_1} + \ldots + e_n x'_{k_n},$$

and as, because of (5),

$$x_0 = x_1 + \ldots + x_n = 0 \quad \text{and} \quad x'_0, x'_1, \ldots, x'_n = 0,$$

it is necessary that

$$e_0 = e_1 = \ldots = e_n;$$

therefore the equalities (6) reduce to the one here:

$$x_i = e x'_{k_i}. \quad (i = 0, 1, 2, \ldots, n; e = \pm 1) \quad (7)$$

The number of substitutions defined by the formulae obtained is equal to $2{\cdot}1{\cdot}2\cdots(n+1)$. By not considering as different two substitutions of opposite coefficients, one will say that the group g is composed of $(n_1)!$ different substitutions. §

With the help of substitutions (7), one can transform any perfect form (2) contiguous to the principal form φ into another form contiguous to the form φ, arbitrarily chosen.

We have arrived at the following important result.

All the perfect forms contiguous to the principal perfect form are equivalent.

31

Let us choose one form among those of the series (2). Let us declare

$$\varphi_1 = \varphi - \rho x_1 x_2.$$

All the perfect form contiguous to the form φ are equivalent to the form φ_1.

Let us find the corresponding value of the parameter ρ.

As we have seen in Number 22, the value searched for of ρ presents the smallest value of the function

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - 1}{x_1 x_2} \tag{8}$$

determined with condition

$$x_1 x_2 > 0.$$

One will distinguish in the subsequent studies two cases:

$$1).\ n = 2 \quad \text{and} \quad 2).\ n \geq 3.$$

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§ See: *Minkowski*, Zur Theorie der positiven quadratischen Formen [On the theory of the positive quadratic forms], (This Journal, V. 101, p. 200)

First case:

By comparing two $n = 2$ binary forms

$$\varphi = x_1^2 + x_2^2 + x_1x_2 \quad \text{and} \quad \varphi_1 = x_1^2 + x_2^2 + x_1x_2 - \rho x_1x_2,$$

one will notice that by making $\rho = 2$ one obtains the form

$$\varphi_1 = x_1^2 + x_2^2 - x_1x_2$$

which is evidently equivalent to the perfect form φ, therefore the perfect form φ_1 is that which one has searched for.

Second case:

By making

$$x_1 = 1,\ x_2 = 1,\ x_3 = -1,\ x_4 = 0, \dots,\ x_n = 0,$$

one obtains a value of the function (8) which is equal to 1.

By making $\rho = 1$, one will present the form φ_1 under the following form:

$$\varphi_1 = \frac{1}{2}\left[(x_1, x_2, \dots, x_n)^2 + (x_1 - x_2)^2 + x_3^2 + \dots + x_n^2\right]. \quad (9)$$

It results in that the form φ_1 is positive. On the ground of that which has been said in Number 23, one will find now all the systems of integers verifying the inequality

$$\varphi_1(x_1, x_2, \dots, x_n) \leq 1.$$

By noticing that the inequality

$$\varphi_1(x_1, x_2, \dots, x_n) < 1$$

is impossible, because the positive form φ_1 has integer values which corresponds to the integer values of variables, one concludes that the form φ_1 is perfect.

With the help of the equality (9), one will easily determine all the presentations of the minimum of the perfect form φ_1.

On the binary and ternary perfect forms and on the domains which correspond to them.

32

The binary principal perfect form

$$\varphi = x_2 + xy + y_2, \quad D = \frac{3}{4}$$

possesses, as we have seen in Number 29, three contiguous perfect forms which are equivalent to the principal form.

One concludes that all the perfect binary forms constitute only a single class of forms equivalent to the principal form.

The domain $\mathcal{R}$ corresponding to the principal form is made up of binary forms (a, b, c) which are determined by the equality

$$ax^2 + 2bxy + cy^2 = \rho x^2 + \rho' y^2 + \rho''(x - y)^2$$

where

$$\rho \geq 0, \;\; \rho' \geq 0, \;\; \rho'' \geq 0$$

It follows that the domain R is determined by the inequalities

$$\rho = a + b \geq 0, \;\; \rho' = -b \geq 0, \;\; \rho'' = c + b \geq 0.$$

By calling reduced the positive binary forms verifying these inequalities, as we have done in Number 27, one will establish a well known method of reduction, due to Mr. Selling. ¶

¶ *Selling.* Über die binären und ternären quadratischen Formen. [On the binary and ternary quadratic forms] (This Journal, V. 77, p. 143)

It results in that the domain R^0 is determined by the inequalities

$$\rho = c - a \geq 0, \ \ \rho' = a + 2b \geq 0, \ \ \rho'' = -b \geq 0.$$

The inequalities obtained only differ from famous conditions of reduction of positive binary quadratic forms due to Lagrange by the choice of the sign of the coefficient b, that which one can arbitrarily make in the method of Lagrange. ‡

33

Let us examine now the ternary perfect forms.

The principal perfect form

$$\varphi = x_2 + y^2 + z^2 + yz + zx + xy, \quad D = \frac{1}{2}$$

possesses six contiguous perfect forms which, all, are equivalent to the perfect form

$$\varphi_1 = x_2 + y^2 + z^2 + yz + zx$$

which we have found in Number 31.

The substitution

$$x = -x', \ y = y', \ z = -y' - z'$$

transforms the form φ_1 into principal form.

One concludes that all the ternary perfect forms form only a single class.

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‡ See: *Lagrange.* Recherches d'Arithmétique. [Studies in arithmetic] (Oeuvres de Lagrange published by Serret, V. III, p. 698)

Gauss. Disquisitiones arithmeticae, art. 171. (Werke, V. I.)

Lejeune Dirichlet. Vorlesungen über Zahlentheorie [Lectures on number theory], published by Dedekind, (Braunschweig 1894, §64, p. 155)

The domain R corresponding to the principal form is made up of all the ternary quadratic forms

$$\begin{pmatrix} a & a' & a'' \\ b & b' & b'' \end{pmatrix}$$

which are determined by the equality

$$ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy =$$
$$\rho_1' + \rho_2 y^2 + \rho_3 z^2 + \rho_4 (y-z)^2 + \rho_5 (z-x)^2 + \rho_6 (x-y)^2.$$

The domain R is determined by the inequalities

$$\rho_1 = a + b' + b'' \geq 0,$$
$$\rho_2 = a' + b'' + b \geq 0,$$
$$\rho_3 = a'' + b + b' \geq 0,$$
$$\rho_4 = -b \geq 0,$$
$$\rho_5 = -b' \geq 0,$$
$$\rho_6 = -b'' \geq 0.$$

By calling reduced the positive ternary quadratic forms belonging to the domain R, one will establish a method of reduction due to Mr. Selling.

The domain R can be partitioned into 24 equivalent parts which can be transformed one into another with the help of 24 substitution adjoined to those which do not change the principal form.

One of these parts, the domain $\mathcal{R}$, will be composed of all the ternary quadratic forms determined by the equality

$$ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy =$$
$$\rho_1 x^2 + \rho_2 y^2 + \rho_3 z^2 + \rho_4 (y-z)^2 + \rho_5 \Psi + \rho_{[6]} \omega$$

where

$$\Psi = x^2 + y^2 + z^2 (y-z)^2 + (z-x)^2,$$
$$\omega = x^2 + y^2 + z^2 (y-z)^2 + (z-x)^2 + (x-y)^2.$$

One will determine the domain $\mathcal{R}$ with the help of inequalities

$$\rho_1 = a + 2b' + b'' \geq 0,$$

$$\rho_2 = a' + b + b' + b'' \geq 0,$$

$$\rho_3 = a'' + b + b' + b'' \geq 0,$$

$$\rho_4 = -b + b' \geq 0,$$

$$\rho_5 = -b' + b'' \geq 0,$$

$$\rho_6 = -b'' \geq 0.$$

The domain $\mathcal{R}$ enjoys the following properties:

1. Any positive ternary quadratic form is equivalent to at least one form belonging to the domain $\mathcal{R}$.

2. Two ternary quadratic forms which are interior to the domain $\mathcal{R}$ can not be equivalent.

By effecting the transformation of the domain $\mathcal{R}$ with the help of all the substitutions of integer coefficients and of determinant ±1, one will make up the set $(\mathcal{R})$ of domains.

Each domain $\mathcal{R}$ belonging to the set $(\mathcal{R})$ possesses six domain contiguous by faces in 5 dimensions.

The domain $\mathcal{R}$ will be transformed into contiguous domain with the help of the following substitutions:

$$S_1 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix},$$

$$S_4 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$S_5 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$S_6 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Each substitution of this series transforms into itself a corresponding face in 5 dimensions of the domain $\mathcal{R}$ and permutes two domains of the set $(\mathcal{R})$ which are contiguous through this face.

This results in a method for the search for the substitution which transforms a given form into a form belonging to the domain $\mathcal{R}$. This method is analogous to that which has been shown in Number 27.

By calling reduced any positive ternary quadratic form belonging to the domain $\mathcal{R}$, one will establish a new method of reduction of positive ternary quadratic forms which can be considered as a generalisation of the method of reduction of Lagrange.

On the perfect form $x_1^2 + x_2^2 + \ldots + x_n^2 + x_1x_3 + x_1x_4 + \ldots + x_{n-1}x_n$.

34

Let us examine the perfect form

$$\varphi_1 = x_1^2 + x_2^2 + \ldots + x_n^2 + x_1x_3 + x_1x_4 + \ldots + x_{n-1}x_n$$

obtained in Number 31. One has admitted

$$a_{ii} = 1,\ (i = 1, 2, \ldots, n),\ \ a_{12} = 0,\ \ a_{ij} = \frac{1}{2}.$$
$$(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

It results in that

$$D = \frac{1}{2^{n-2}}.$$

By supposing that $n \geq 4$, one will have $n^2 - n$ representations of the minimum of the form φ_1 the number of which is greater than $\frac{n(n+1)}{2}$.

These representations of the minimum of the form φ_1 will be characterised by the linear forms

$$\begin{cases} \lambda_1 = x_1, \lambda_2 = x_2, \dots, \lambda_n = x_n, \\ \lambda_{n+1} = x_1 - x_3, \dots, \\ \lambda_{\frac{n(n+1)}{2}-1} = x_{n-1} - x_n,\ \lambda_{\frac{n(n+1)}{2}} = x_1 + x_2 - x_3, \dots, \\ \lambda_{\frac{n(n+1)}{2}+n-3} = x_1 + x_2 - x_n, \\ \lambda_{\frac{n(n+1)}{2}+n-2} = x_1 + x_2 - x_3 - x_4, \dots, \\ \lambda_{n^2-n} = x_1 + x_2 - x_{n-1} - x_n. \end{cases} \tag{1}$$

The domain R_1 corresponding to the perfect form φ_1 is made up of forms determined by the equality

$$f(x_1, x_2, \dots, x_n) = \sum_{k=1}^{n^2-n} \rho_k \lambda_k^2 \text{ where } \rho_k \geq 0.$$

$$(k = 1, 2, \dots, [n^2 - n])$$

Let us find the linear inequalities which define the domain R_1.

The number of these inequalities is so large in deed for $n = 4$.

One will overcome the difficulties which result by the help of a particular method.

35

Let us find the group g_1 of substitutions which do not change the form φ_1.

To this effect, let us introduce in our studies a quadratic form ω determined by the equality

$$\omega = \frac{2}{n-1}(\lambda_1^2 + \lambda_2^2 + \dots + \lambda_{n^2-n}^2).$$

After the reductions, one obtains

$$\begin{aligned}\omega(x_1, x_2, \ldots, x_n) =& nx_1^2 + nx_2^2 + 4x_3^2 + \ldots \\ & + 4x_n^2 + 2(n-2)x_1x_2 - 4x_1x_3 - \ldots \\ & - 4x_1x_n - 4x_2x_3 - \ldots - 4x_2x_n.\end{aligned}$$

One can give in the form ω_2 the following expression:

$$\begin{aligned}\omega(x_1, x_2, \ldots, x_n) =& (x_1 - x_2)^2 + (x_1 + x_2)^2 + \\ & (x_1 + x_2 - 2x_3)^2 + \ldots + (x_1 + x_2 - 2x_n)^2.\end{aligned}$$

It is easy to demonstrate that the form added to the perfect form φ_1 has coefficients which are proportional to those of the form ω.

It follows that the perfect form φ_1 is extreme.

Let us observe that the linear form

$$x_1 + x_2 + \ldots + x_n,\ x_1 - x_2,\ x_3,\ x_4, \ldots,\ x_n$$

characterise n minimum 4 representations of the form ω. In the case $n \geq 5$, other representations of the minimum of the form ω do not exist; in the case $n = 4$, one obtains 12 representations of the minimum of the form ω.

By noticing that

$$\varphi_1 = \frac{1}{2}\left[(x_1 + x_2 + \ldots + x_n)^2 + (x_1 - x_2)^2 + x_3^2 + \ldots + x_n^2\right],$$

one can say that the group g_1, in the case $n \geq 5$, is composed of all the permutations of the forms

$$(x_1 + x_2 + \ldots + x_n)^2,\ (x_1 - x_2)^2,\ x_3^2, \ldots,\ x_n^2.$$

In the case $n = 4$, one will determine by this method only divisor of the group g_1.

By indicating

$$u_1 = x_1 + x^2 + \ldots + x_n,$$

$$u_2 = x_1 - x_2,$$

$$u_3 = x_3, \ldots, u_n = x_n,$$

$$u'_1 = x'_1 + x'_2 + \ldots + x'_n,$$

$$u'_2 = x'_1 - x'_2,$$

$$u'_3 = x'_3, \ldots, u'_n = x'_n$$

let us declare

$$u_i = e_i u'_{k_i}, \quad (i = 1, 2, \ldots, n) \tag{2}$$

where $e_i = \pm 1$ $(i = 1, 2, \ldots, n)$ and the indices $k_1, k_2, \ldots, k_n$ present any one permutation of numbers $1, 2, \ldots, n$.

Each system of equalities (2) defines a substitution of the group g_1.

One concludes that the group g_1 is composed of - $2^{n-1}n!$ different substitutions, in the case $n \geq 5$.

36

Let us suppose that the domain R_1 be determined by the inequalities

$$\sum p_{ij}^{(k)} a_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

By indicating

$$\Psi_k(x_1, x_2, \ldots, x_n) = \sum p_{ij}^{(k)} x_i x_j, \quad (k = 1, 2, \ldots, \sigma)$$

one will determine, as we have seen in Number 22, σ perfect forms

$$\varphi_1^{(k)} = \varphi_1 + \rho_k \Psi_k \quad (k = 1, 2, \ldots, \sigma) \tag{3}$$

contiguous to the perfect form φ_1.

All the substitutions of the group g_1 will make only one permutation of forms

$$\Psi_1, \Psi_2, \ldots, \Psi_\sigma. \tag{4}$$

Let us effect the transformation of forms (3) and (4) with the help of the substitution

$$x_1, x_2, \ldots, x_n = x'_1,\ x_1 - x_2 = x'_2,\ x_3 = x'_3, \ldots,\ x_n = x'_n. \quad (5)$$

The series (4) will be transformed into a series

$$\Psi'_1, \Psi'_2, \ldots, \Psi'_\sigma.$$

Let us indicate by **g** a group of substitutions

$$x'_i = e_i x''_{k_i}, \quad (i = 1, 2, \ldots, n)$$

where $e = \pm 1$ $(i = 1, 2, \ldots, n)$ and $k_1, k_2, \ldots, k_n$ present a permutation of numbers $1, 2, \ldots, n$. Each substitution of the group **g** makes only one permutation of forms (6), and to a similar substitution corresponds a substitution of the group g_1.

By indicating

$$\Psi'_k(x'_1, x'_2, \ldots, x'_n) = \sum P_{ij}^{(k)} x'_i x'_j, \quad (k = 1, 2, \ldots, \sigma)$$

one will determine with the help of inequalities

$$\sum P_{ij}^{(k)} a_i a_j \geq 0, \quad (k = 1, 2, \ldots, \sigma) \quad (7)$$

a domain $\mathcal{R}$.

The form φ_1 will be transformed into a form

$$\frac{1}{2}({x'_1}^2 + {x'_2}^2 + \ldots + {x'_n}^2),$$

with the help of the substitution (5), and any system $(x_1, x_2, \ldots, x_n)$ of integers $x_1, x_2, \ldots, x_n$ will be replaced by a system $(x'_1, x'_2, \ldots, x'_n)$ of number, also integer, $x'_1, x'_2, \ldots, x'_n$ satisfying the condition

$$x'_1 + x'_2 + \ldots + x'_n \equiv 0 (\mathrm{mod}\ 2). \quad (8)$$

It results in that the linear forms (1) which correspond to the various representations of the minimum of the form φ_1 will be replaced by the forms

$$x'_i + x'_j \text{ and } x'_i - x'_j \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

which characterise the various representations of the minimum 2 of the quadratic form ${x'_1}^2 + {x'_2}^2 + \ldots + {x'_n}^2$, in the set (X') of all the systems $(x'_1, x'_2, \ldots, x'_n)$ of integers $x'_1, x'_2, \ldots, x'_n$ satisfying the condition (8).

One concludes that the edges of the domain $\mathcal{R}$ will be characterised by the quadratic form

$$(x'_i + x'_j)^2 \quad \text{and} \quad (x'_i - x'_j)^2.$$

$$(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

By virtue of (7), one obtains the inequalities

$$P^{(k)}_{ii} + 2P^{(k)}_{ij} + P^{(k)}_{jj} \geq 0 \text{ and } P^{(k)}_{ii} - 2P^{(k)}_{ij} + P^{(k)}_{jj} \geq 0.$$

$$(k = 1, 2, \ldots, \sigma; i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j) \tag{9}$$

Let us examine any one form

$$\Psi'(x'_1, x'_2, \ldots, x'_n) = \sum P_{ij} x'_i x'_j \tag{10}$$

belonging to the series (6). By virtue of (9), one will have

$$P_{ii} + 2P_{ij} + P_{jj} \geq 0 \text{ and } P_{ii} - 2P_{ij} + P_{jj} \geq 0.$$

$$(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j) \tag{11}$$

Among these conditions one will find t quantities which define the coefficients of the form (10) to an immediate common factor. All these equalities will be of the form

$$P_{kk} - 2e_{kh}P_{kh} + P_{hh} = 0 \text{ where } e_{kh} = \pm 1. \tag{12}$$

Let us suppose that there exists a combination of values of k and h satisfying the conditions

$$P_{kk} + 2P_{kh} + P_{hh} > 0 \text{ and } P_{kk} - 2P_{kh} + P_{hh} > 0. \tag{13}$$

By noticing that the coefficient P_{kh} does not enter the other inequalities (11), one concludes that the coefficient P_{kh} remains undetermined.

For all the coefficients of the form (10) to be determined by the conditions (12) to an immediate common factor, it is necessary, the coefficient P_{kh} being independent of other coefficients, that all the coefficients which remain cancel out.

By virtue of inequalities (13), this supposition is impossible, therefore the inequality (12) has to hold for all the values of indices k and h.

One obtains $\frac{n(n-1)}{2}$ conditions

$$P_{kk} - 2e_{kh}P_{kh} + P_{hh} = 0 \text{ where } e_{kh} = \pm 1.$$

$$(k = 1, 2, \ldots, n; h = 1, 2, \ldots, n; k \neq h) \tag{14}$$

which serve to determine the coefficients P_{kh} in functions of coefficients

$$P_{11}, P_{22}, \ldots, P_{nn}. \tag{15}$$

The coefficients $P_{11}, P_{22}, \ldots, P_{nn}$ can not be independent, and will be connected by at least $n-1$ equations of the form (12). Therefore, in at least $n-1$ case, one will have the equations of the form

$$P_{kk} \pm 2P_{kh} + P_{hh} = 0. \tag{16}$$

To make short we will call these equations double.

This stated, let us suppose, in the first place, that there exists at least one coefficient among those of the series (15) which does not enter in the double equations (16). One can suppose, to fix the ideas, that P_{11} be such a coefficient.

The coefficient P_{11} being independent, all the coefficients $P_{22}, \ldots, P_{nn}$ will cancel each other and, by virtue of (14), the coefficients

$$P_{23}, P_{24}, \ldots, P_{n-1,n}$$

will also cancel one another out.

The coefficient P_{11} is used for determining the coefficients P_{12}, $P_{13}, \ldots, P_{1n}$ with the help of equations (14) which take the form

$$P_{11} - 2e_{1k}P_{1k} = 0; \quad (k = 2, 3, \ldots, n)$$

it follows that

$$2P_{1k} = e_{1k}P_{11}. \quad (k = 2, 3, \ldots, n)$$

As, on the ground of the supposition made,

$$P_{11} + 2e_{1k}P_{1k} > 0, \quad (k = 2, 3, \ldots, n)$$

it is necessary that

$$P_{11} > 0,$$

and one can declare

$$P_{11} = 1.$$

The form (10) is determined by the equalities obtained, and one will have

$$\Psi'(x'_1, x'_2, \ldots, x'_n) = {x'_1}^2 + e_{12}x'_1x'_2 + \ldots + e_{1n}x'_1x'_n. \qquad (17)$$

By replacing the variables

$$e_{12}x'_2, \ e_{13}x'_3, \ldots, \ e_{1n}x'_n$$

by the variables $x'_2, \ldots, x'_n$, one will replace the form (17) by the form

$$\Psi'(x'_1, x'_2, \ldots, x'_n) = x'_1(x'_1, x'_2, \ldots, x'_n).$$

Let us suppose, in the second place, that all the coefficients (15) enter in the double equations (16).

At least one of the coefficients (15) is not zero. Let us suppose that $P_{kk} \neq 0$. Following the hypothesis, the coefficient P_{kk} enters in at least one double equation

$$P_{kk} \pm 2P_{kh} + P_{hh} = 0.$$

It follows that

$$P_{kh} = 0 \text{ and } P_{kk} + P_{hh} = 0,$$

therefore the coefficients P_{kk} and P_{hh} are of opposite signs. Let us suppose, to fix the ideas, that

$$P_{11} = -1. \qquad (18)$$

By examining the inequalities

$$P_{11} \pm 2P_{1k} + Pkk \geq 0, \quad (k = 2, 3, \ldots, n)$$

one deduces

$$Pkk > 0. \quad (k = 2, 3, \ldots, n)$$

It results in that the double equation

$$P_{kk} \pm 2P_{kh} + P_{hh} = 0$$

has to be impossible so long as $k \geq 2$ and $h \geq 2$, therefore all the double equations will be of the form

$$P_{11} \pm 2P_{1k} + P_{kk} = 0. \quad (k = 2, 3, \ldots, n)$$

From these equations one gets, by virtue of (18),

$$P_{kk} = 1 \text{ and } P_{1k} = 0. \quad (k = 2, 3, \ldots, n) \qquad (19)$$

By substituting the values obtained of coefficients $P_{11}, P_{22}, \ldots, P_{nn}$ in the equations

$$P_{kk} - 2e_{kh}P_{kh} + P_{hh} = 0 \text{ where } e_{kh} = \pm 1,$$

$$(k = 2, 3, \ldots, n; h = 2, 3, \ldots, n; k \neq h)$$

one obtains, because of (19),

$$P_{kh} = e_{kh} \quad \text{where} \quad e_{kh} = \pm 1.$$

$$(k = 2, 3, \ldots, n; h = 2, 3, \ldots, n; k \neq h)$$

The form (10) will have for expression

$$\begin{aligned}\Psi'(x'_1, x'_2, \ldots, x'_n) = -{x'_1}^2 + {x'_2}^2 + {x'_3}^2 + \ldots \\ + {x'_n}^2 + 2e_{23}x'_2x'_3 + 2e_{24}x'_2x'_4 + \ldots \\ + 2e_{n-1,n}x'_{n-1}x'_n\end{aligned} \tag{20}$$

where

$$e_{23} = \pm 1, \ e_{24} = \pm 1, \ldots, \ e_{n-1,n} = \pm 1.$$

One obtains in this way $2^{\frac{(n-1)(n-2)}{2}}$ different forms. By permuting the variables and by changing their signs, one will particularly decrease the number of various forms determined by the formula (20).

37

With the help of results obtained, one can easily recognise whether a given quadratic form $\sum a_{ij}x_ix_j$ belongs to the domain $\mathcal{R}$ or not.

One will examine, in the first place, the sums

$$e_{1k}a_{1k} + e_{2k}a_{2k} + \ldots + e_{nk}a_{nk}$$

where $e_{1k} = \pm 1, e_{2k} = \pm 1, \ldots, e_{nk} = \pm 1$ and $e_{kk} = 1$. $(k = 1, 2, \ldots, n)$

All these sums have to be positive or zero. The inequalities

$$a_{kk} - |a_{1k}| - \ldots - |a_{k-1,k}| - |a_{k+1,k}| - \ldots - |a_{nk}| \geq 0,$$

$$(k = 1, 2, \ldots, n) \tag{21}$$

present the conditions necessary and sufficient for the inequalities

$$e_{1k}a_{1k} + e_{2k}a_{2k} + \ldots + e_{nk}a_{nk} \geq 0$$

$$(k = 1, 2, \dots, n)$$

to hold.

Let us examine, in the second place, the inequalities

$$-a_{11} + a_{22} + a_{33} + \dots + a_{nn} + 2e_{23}a_{23} + 2e_{24}a_{24} + \dots$$
$$+2e_{n-1,n}a_{n-1,n} \geq 0$$

where

$$e_{23} = \pm 1,\ \ e_{24} = \pm 1, \dots,\ \ e_{n-1,n} = \pm 1,$$

These inequalities can be replaced by a single one

$$-a_{11} + a_{22} + a_{33} + \dots + a_{nn} - 2|a_{23}| - 2|a_{24}| - \dots - 2|a_{n-1,n}| \geq 0.$$

One will present this inequality under the form

$$a_{11} + a_{22} + \dots + a_{nn} - 2|a_{12}| - 2|a_{13}| - \dots - 2|a_{n-1,n}|$$
$$\geq 2\left(a_{11} - |a_{12}| - \dots - |a_{1n}|\right).$$

By permuting the variables, one obtains n inequalities

$$a_{11} + a_{22} + \dots + a_{nn} - 2|a_{12}| - 2|a_{13}| - \dots - 2|a_{n-1,n}|$$
$$\geq 2\left(a_{kk} - |a_{1k}| - \dots - |a_{nk}|\right) \text{ where } k = 1, 2, \dots, n \tag{22}$$

We have arrived at the following result. One can easily recognise whether a given positive quadratic form f belong to the domain R_1 or not. To this effect, one will transform the form f by a form f' with the help of the substitution adjoint to the substitution (5) and one will examine $2n$ inequalities (21) and (22). For the form f to belong to the domain R_1, it is necessary and sufficient that the form f' verifies $2n$ inequalities (21) and (22).

38

Let us return now to the perfect forms (3) contiguous to the perfect form φ_1. We have seen that these forms will be transformed with the help of the substitution (5) into forms

$$\frac{1}{2}\left({x'_1}^2 + {x'_2}^2 + \ldots + {x'_n}^2\right) + \rho_k \Psi'_k(x'_1, x'_2, \ldots, x'_n).$$
$$(k = 1, 2, \ldots, \sigma)$$

The forms $\Psi'_1, \Psi'_2, \ldots, \Psi'_\sigma$ can be transformed with the help of substitutions belonging to the group $\mathbf{g}$ into forms

$$\left\{\begin{array}{l} 1).\ x'_3(-x'_1 - x'_2 + x'_3 + x'_4 + \ldots + x'_n), \\ 2).\ \quad - {x'_2}^2 + {x'_1}^2 + {x'_3}^2 + \ldots + {x'_n}^2 - 2x'_1x'_3 - \ldots \\ \qquad\quad - 2x'_1x'_n + 2e_{34}x'_3x'_4 + \ldots + 2e_{n-1,n}x'_{n-1}x'_n, \end{array}\right. \tag{23}$$

where

$$e_{34} = \pm 1, \ldots,\ e_{n-1,n} = \pm 1.$$

The inverse substitution to the substitution (5):

$$x'_1 = x_1 + x_2 + \ldots + x_n,\ x'_2 = x_1 - x_2,\ x'_3 - x_3, \ldots,\ x'_n = x_n$$

will transform the forms (23) into forms

$$\begin{array}{ll} 1). & -2x_1x_3 \\ 2). & 4(x_1x_2 - \delta_{34}x_3x_4 - \ldots - \delta_{n-1,n}x_{n-1}x_n), \\ & \quad \text{where; } \delta_{34} = 0 \text{ or } 1, \ldots, \delta_{n-1,n} = 0 \text{ or } 1. \end{array}$$

One concludes that all the perfect forms contiguous to the form φ_1 are equivalent to the following perfect forms

$$\begin{array}{l} 1).\varphi_1 - \rho x_1x_3, \\ 2).\varphi_1 + \rho(x_1x_2 - \delta_{34}x_3x_4 - \ldots - \delta_{n-1,n}x_{n-1}x_n), \end{array}$$

where

$$\delta_{34} = 0 \text{ or } 1, \ldots,\ \delta_{n-1,n} = 0 \text{ or } 1.$$

39 *Study of the perfect form $\varphi_1 - \rho x_1 x_3$.*

The perfect form φ_1, possesses, as we have seen in Number 38, many contiguous perfect forms which are not equivalent.

One will determine in the following only a single perfect form

$$\varphi_2 = \varphi_1 - \rho x_1 x_3$$

contiguous to the perfect form φ_1.

We have demonstrated in Number 22 that the parameter ρ presents the smallest value of the function

$$\rho = \varphi_1 \frac{(x_1, x_2, \ldots, x_n) - 1}{x_1 x_3} \tag{1}$$

determined on condition that

$$x_1 x_3 > 0. \tag{2}$$

By declaring

$$x_1 = 1,\ x_2 = 0,\ x_3 = 1,\ x_4 = -1,\ x_5 = 0, \ldots,\ x_n = 0,$$

one obtains the value of the function (1) which is equal to 1, therefore

$$0 < \rho \leq 1. \tag{3}$$

Let us effect the transformation of the function (1) with the help of the substitution

$$\begin{gathered} x_3 = x'_1, \\ -x_1 + x_2 = x'_2, \\ x_1 + x_2 + \ldots + x_n = -x'_3, \\ x_4 = x'_4, \ldots,\ x_n = x'_n \end{gathered} \tag{4}$$

one will have

$$\rho = \frac{{x'_1}^2 + {x'_2}^2 + \ldots + {x'_n}^2 - 2}{-x'_1(x'_1 + x'_2 + \ldots + x'_n)} \tag{5}$$

where, because of (2),

$$x'_1(x'_1 + x'_2 + \ldots + x'_n) < 0 \tag{6}$$

and, because of (4),

$$x'_1 + x'_2 + \ldots + x'_n \equiv 0(\bmod 2),$$

the variables $x'_1 + x'_2 + \ldots + x'_n$ being integers.

Let us indicate

$$f(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2 + \rho x_1(x_1 + x_2 + \ldots + x_n).$$

By virtue of (5) and (6) the value looked for of ρ is defined by the conditions that the inequality

$$f(x_1, x_2, \ldots, x_n) < 2$$

is impossible, so long as the integers $x_1, x_2, \ldots, x_n$ verify the congruence

$$x_1 + x_2 + \ldots + x_n \equiv 0(\bmod 2), \tag{7}$$

and that there exists at least one system $(l_1, l_2, \ldots, l_n)$ verifying the equation

$$f(x_1, x_2, \ldots, x_n) = 2 \tag{8}$$

and the congruence (7).

The form f can be determined by the equality

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) = \left(x_2 + \rho\frac{x_1}{2}\right)^2 + \left(x_3 + \rho\frac{x_1}{2}\right)^2 + \ldots \\ + \left(x_n + \rho\frac{x_1}{2}\right)^2 + \left(1 + \rho - \frac{n-1}{4}\rho^2\right) x_1^2. \end{aligned} \tag{9}$$

It follows that the form f will be positive, provided that

$$1 + \rho - \frac{n-1}{4}\rho^2 > 0,$$

and the upper limit R of values of ρ verifies the equation

$$1 + R - \frac{n-1}{4}R^2 = 1,$$

therefore

$$\frac{R = 2}{\sqrt{n} - 1.} \tag{10}$$

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This presented, let us examine a system $(l_1, l_2, \ldots, l_n)$ of integers verifying the equation (8) and the congruence (7).

I say that there will be the inequalities

$$\left| l_i + \rho \frac{l_1}{2} \right| \leq 1. \quad (i = 2, \ldots, n) \tag{11}$$

In effect, if one suppose that

$$\left| l_k + \rho \frac{l_1}{2} \right| < 1,$$

one will determine $e_k = \pm 1$ such that the inequality

$$\left| l_k + 2e_k + \frac{l_1}{2} \right| < \left| l_k + \rho \frac{l_1}{2} \right|$$

holds, and one will present

$$l'_i = l_i \text{ and } l'_k = l_k = 2e_k. \quad (i = 1, 2, \ldots, n; i \neq k)$$

The condition (7) will be satisfied, and one will have, by virtue of (9), the inequality

$$f(l'_1, l'_2, \ldots, l'_n) < 2,$$

which is contrary to the hypothesis.

By examining the inequalities (11) and the form f with the help of the formula (9), one will easily demonstrate that among the system of integers verifying the equation (8) with condition (7) is found at least one system $(l_1, l_2, \ldots, l_n)$ satisfying the conditions

$$f(l_1, l_2, \ldots, l_n) = 2 \tag{12}$$

and

$$l_2 = l_3 + \delta,\ l_3 = l_4 = \ldots = l_n \text{ where } \delta = 0 \text{ or } \pm 1. \tag{13}$$

By virtue of (6), one will have the inequality

$$l_1 \left[l_1 + \delta + (n-2) l_3 \right] < 0.$$

One can suppose that

$$l_1 < 0, \tag{14}$$

and it follows that

$$l_1 + \delta + (n-2) l_3 > 0,$$

therefore, because of (13) and (14), it is necessary that

$$l_3 > 0.$$

I say that $l_3 = 1$. To demonstrate this, let us effect the transformation of the positive quadratic form $f(x_1, x_2, \ldots, x_n)$ with the help of the substitution

$$x_1 = -x,\ x_2 = y,\ x_3 = x_4 = \ldots = x_n - z; \tag{15}$$

one will obtain a ternary positive form

$$F(x, y, z) = x^2 + y^2 + (n-2) z^2 - \rho x(-x + y + (n-2) z).$$

By virtue of the condition (7), the integers x, y, z verify the congruence

$$x + y + (n-2) z \equiv 0 (\mathrm{mod}\ 2). \tag{16}$$

By indicating

$$u = -l_1, \quad v = l_2, \quad w = l_3,$$

one will have, because of (12), (13) and (15),

$$F(u, v, w) = 2,$$

and the condition (16) will be fulfilled.

The inequality

$$f(x, y, z) < 2$$

is impossible so long as the integers x, y, z verify the congruence (16).

Let us effect the transformation of the form $F(x, y, z)$ with the help of the substitution

$$x = x_{,} + y' + (n-2)z', \ \ y = x' - y', \ \ z = z'. \qquad (17)$$

The set of systems (x, y, z) of integers verifying the congruence (16) will be replaced by the set of systems (x', y', z') of arbitrary integers.

Let us indicate by $F'(x', y', z')$ the transformed form. Let D and D' be the determinants of forms $F(x, y, z)$ and $F'(x', y', z')$. By virtue of (17), one will have

$$D' = 4D. \qquad (18)$$

Let us notice that the number 2 presents the minimum of the form obtained $F'(x', y', z')$ determined in the set of all the systems (x', y', z') of integers, the system $(0, 0, 0)$ being excluded.

On the ground of the known theorem § on the limit of the minimum of a ternary positive quadratic form, one will have the inequality

$$2 \leq \sqrt[3]{2D'}.$$

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§ See: *Gauss.* Werke, V. II, p. 192, Göttingen 1863.

Lejeune-Dirichlet. Über die Reduktion der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen. (This Journal, V. 40, p. 209)

Hermite. Sur la théorie des formes quadratiques ternaires. [On the theory of ternary quadratic forms] (This Journal, V. 40, p. 173)

It follows that

$$D' \geq 4,$$

and because of (18), one obtains

$$D \geq 1. \tag{19}$$

This presented, let us observe that the form $F(x, y, z)$ has the following values:

$$F(u, v, w) = 2, \;\; F(1, 1, 0) = 2, \;\; F(1, -1, 0) = 2 + 2\rho.$$

By transforming the form $F(x, y, z)$ with the help of the substitution

$$\begin{pmatrix} u, & 1, & 1 \\ v, & 1, & -1 \\ w, & 0, & 0 \end{pmatrix}, \tag{20}$$

one obtains a form

$$F_0(x', y', z') = ax'^2 + a'y'^2 + a''z'^2 + 2by'z' + 2b'z'x' + 2b''x'y',$$

where

$$a = 2, \; a' = 2, \; a'' = 2 + 2\rho \text{ and } b = \rho. \tag{21}$$

The product $a{\cdot}a'{\cdot}a''$ in any positive ternary quadratic form $\begin{pmatrix} a, & a', & a'' \\ b, & b' & b'' \end{pmatrix}$ is, as one knows, always greater than the determinant of the form, unless the coefficients b, b', b'' do not simultaneously cancel one another out.

By indicating with D_0 the determinant of the form $F_0(x', y', z')$, one will have, because of (21),

$$D_0 < 4(2 + 2\rho),$$

and as, by virtue of (20),

$$D_0 = 4w^2 D,$$

it becomes

$$w^2 D < 2 + 2\rho.$$

By virtue of (3) and (19), one obtains the inequality

$$w^2 < 4,$$

therefore

$$w = 1.$$

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By returning to the equalities (13), one obtains

$$l_1 = -u, \; l_2 = \delta \text{ and } l_3 = 1, \; l_4 = 1, \ldots, \; l_n = 1,$$

where

$$u > 0 \quad \text{and} \quad \delta = 0, 1, 2.$$

By substituting the values found of $l_1, l_2, \ldots, l_n$ in the function (5), one will have

$$\rho = \frac{u^2 + \delta^2 + n - 4}{u(-u + \delta + n - 2)} \tag{22}$$

It remains to determine the smallest value of this function providing that

$$u > 0, \; -u + \delta + n - 2 > 0, \; u \equiv n + \delta (\mathrm{mod}\, 2) \text{ and } \delta = 0, 1, 2. \tag{23}$$

Let us admit

$$u = \sqrt{n} - 1 + \alpha, \tag{24}$$

α being a real number.

The function (22) takes the form

$$\rho = \frac{2n + (2\alpha - 2)\sqrt{n} + \alpha^2 - 2\alpha + \delta^2 - 3}{\sqrt[n]{n} + (\alpha - 2)n + (\delta - 2\alpha)\sqrt{n} + 1 - \alpha^2 + \alpha\delta - \delta}.$$

The value searched for of ρ has to verify the inequality

$$\rho < R,$$

therefore because of (10), one will have

$$\frac{2}{\sqrt{n}-1} - \rho > 0. \tag{25}$$

After the reductions, one obtains

$$\frac{2}{\sqrt{n}-1} - \rho =$$

$$\frac{(1-\delta^2+2\delta-\alpha^2)\sqrt{n}+\delta^2-2\delta-1-\alpha^2-2\alpha+2\alpha\delta}{(\sqrt{n}-1)\left[n\sqrt{n}+(\alpha-2)n+(\delta-2\alpha)\sqrt{n}+1-\alpha^2+\alpha\delta-\delta\right]}$$

and, because of (25), it becomes

$$(1-\delta^2+2\delta-\alpha^2)\sqrt{n}+\delta^2-2\delta-1-\alpha^2-2\alpha+2\alpha\delta > 0.$$

By noticing that

$$\delta^2-2\delta-1-\alpha^2-2\alpha+2\alpha\delta \leq 0 \quad \text{so long as} \quad \delta = 0,1,2,$$

one obtains the inequality

$$1-\delta^2+2\delta-\alpha^2 > 0.$$

By making $\delta = 0$ and 2, one will have

$$\alpha^2 < 1 \text{ as long as } \delta = 0 \text{ and } 2 \tag{26}$$

By making $\delta = 1$, one will have

$$\alpha^2 < 2 \quad \text{so long as} \quad \delta = 1. \tag{27}$$

Let us indicate by m a positive integer determined with the help of inequalities

$$\sqrt{n}-1 \leq m < \sqrt{n}. \tag{28}$$

By declaring

$$n = m^2 + p, \tag{29}$$

one will have a positive integer p verifying the inequalities

$$0 < p \leq 2m + 1. \tag{30}$$

First case: p is an odd number.

By virtue of (23) and (29), one will have a congruence

$$u \equiv m^2 + p + \delta (\operatorname{mod} 2),$$

p being an odd number; one can declare

$$u = m^2 + \delta + 1 + 2t. \tag{31}$$

By declaring

$$\sqrt{n} = m + \xi,$$

one will have

$$0 < \xi \leq 1, \tag{32}$$

because of (28). By virtue of (24), one obtains the equality

$$u = m - 1 + \xi + \alpha,$$

and because of (31), it becomes

$$\xi + \alpha = m^2 - m + 2 + 2t + \delta. \tag{33}$$

By supposing that $\delta = 0$ or 2, one obtains

$$\xi + \alpha \equiv 0 (\operatorname{mod} 2).$$

By virtue of (26) and (32), it is necessary that

$$\xi + \alpha = 0,$$

therefore

$$u = m - 1 \text{ so long as } \delta = 0 \text{ and } \delta = 2.$$

By supposing that $\delta = 1$, one obtains, because of (33),

$$\xi + \alpha \equiv 1 (\operatorname{mod} 2).$$

By virtue of (27) and (32), the integer $\xi + \alpha$ can have only two values

$$\xi + \alpha = \pm 1,$$

and it results in that

$$u = m \quad \text{or} \quad m - 2 \quad \text{as long as} \quad \delta = 1.$$

One obtains four values of the function (22):

$$\rho_1 = \tfrac{(m-1)^2+n-4}{(m-1)(n-m-1)}, \qquad \rho_2 = \tfrac{m^2+n-3}{m(n-m-1)},$$
$$\rho_3 = \tfrac{(m-2)^2+n-3}{(m-2)(n-m+1)}, \quad \rho_4 = \tfrac{(m-2)^2+n}{(m-1)(n-m+1)}$$

among which is found the smallest value looked for of ρ.

By noticing that

$$\rho_1 - \rho_2 = \frac{p-3}{m(m-1)(n-m-1)},$$
$$\rho_4 - \rho_1 = \frac{2p+2}{(m-1)(n-m-1)(n-m+1)},$$
$$\rho_3 - \rho_4 = \frac{p+1}{(m-1)(m-2)(n-m+1)},$$

one obtains, because of (30),

$$\rho_1 < \rho_4 < \rho_3$$

and

$$\rho_2 \leq \rho_1 \quad \text{so long as} \quad p \geq 3,$$
$$\rho_1 < \rho_2 \quad \text{so long as} \quad p < 3.$$

There exists only a single odd value of p verifying the inequalities $0 < p < 3$, therefore one will have the inequality

$$\rho_1 < \rho_2 \quad \text{so long as} \quad p = 1.$$

We have arrived at the following result.

The smallest value of ρ *will have for expression*

$$\rho = \frac{m^2+n-3}{m(n-m-1),} \tag{34}$$

provided that $n = m^2 + p$, *and the odd number* p *verifies the inequalities*

$$3 \leq p \leq 2m + 1.$$

In the case $n = m^2 + 1$, *the smallest value of* ρ *will be*

$$\rho = \frac{(m-1)^2 + n - 4}{(m-1)(n-m-1)}.$$

Second case: p is an even number.

One will have, because of (23), the inequality [sic]

$$u \equiv m^2 + \delta (\mathrm{mod}\ 2).$$

By presenting

$$u = m^2 + \delta + 2t,$$

one will have the equalities

$$u = m - 1 + \xi + \alpha \quad \text{and} \quad \xi + \alpha = m^2 - m + 2t + \delta + 1.$$

By supposing that $\delta = 0$ or 2, one obtains

$$\xi + \alpha = 1,$$

and it follows that

$$u = m \quad \text{so long as} \quad \delta = 0 \quad \text{and} \quad 2.$$

By supposing that $\delta = 1$, one obtains

$$\xi + \alpha = 0 \quad \text{or} \quad \xi + \alpha = 2,$$

therefore

$$u = m + 1 \quad \text{or} \quad u = m + 1 \quad \text{so long as} \quad \delta = 1.$$

The smallest value of ρ is found among the following values of the function (22):

$$\rho_1 = \tfrac{m^2+n-4}{m(n-m-2)}, \qquad \rho_2 = \tfrac{(m+1)^2+n-3}{(m+1)(n-m-2)},$$
$$\rho_3 = \tfrac{(m-1)^2+n-3}{(m-1)(n-m)}, \qquad \rho_4 = \tfrac{m^2+n}{m(n-m)}.$$

By noticing that

$$\begin{aligned}\rho_2 - \rho_1 &= \frac{2m+4-p}{m(m+1)(n-m-2)},\\ \rho_1 - \rho_4 &= \frac{4m-2p}{m(n-m)(n-m-2)},\\ \rho_4 - \rho_3 &= \frac{2m-p}{m(m-1)(n-m)},\end{aligned}$$

one obtains, because of (30),

$$\rho_3 \le \rho_4 \le \rho_1 < \rho_2.$$

We have arrived at the following result:

The smallest value of ρ *is expressed by the equality*

$$\rho = \frac{(m-1)^2+n-3}{(m-1)(n-m)}$$

provided that $n = m^2 + p$, *and the even number* p *verifies the inequalities*

$$0 < p < 2m+1.$$

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We have determined the value of the parameter ρ which defines the perfect form $\varphi_1 + \rho x_1 x_3$. The determinant D of this form, by virtue of (4) and (9), will have for expression

$$D = \frac{4+4\rho-(n-1)\rho^2}{2^n}. \tag{35}$$

The corresponding value of the function $\mathcal{M}(a_{ij})$ defined in Number 16 will be

$$\mathcal{M}(a_{ij}) = 2\sqrt[n]{\frac{1}{4+4\rho-(n-1)\rho^2}}.$$

By applying the formulae obtained to the case:

$$n = 4, 5, 6, 7, 8,$$

one obtains the same value of ρ

$$\rho = 1.$$

The corresponding perfect forms will be

$$x_1^2 + x_2^2 + \ldots, x_4^2 + x_1x_4 + \ldots + x_3x_4,\ D = \frac{5}{2^4},\ \mathcal{M}(a_{ij}) = 2\sqrt[4]{\frac{1}{5}};$$

$$x_1^2 + x_2^2 + \ldots, x_5^2 + x_1x_4 + \ldots + x_4x_5,\ D = \frac{4}{2^5},\ \mathcal{M}(a_{ij}) = 2\sqrt[5]{\frac{1}{4}};$$

$$x_1^2 + x_2^2 + \ldots, x_6^2 + x_1x_4 + \ldots + x_5x_6,\ D = \frac{3}{2^6},\ \mathcal{M}(a_{ij}) = 2\sqrt[6]{\frac{1}{3}};$$

$$x_1^2 + x_2^2 + \ldots, x_7^2 + x_1x_4 + \ldots + x_6x_7,\ D = \frac{2}{2^7},\ \mathcal{M}(a_{ij}) = 2\sqrt[7]{\frac{1}{2}};$$

$$x_1^2 + x_2^2 + \ldots, x_8^2 + x_1x_4 + \ldots + x_7x_8,\ D = \frac{1}{2^8},\ \mathcal{M}(a_{ij}) = 2.$$

One comes across all these perfect forms in the Mémoire of Mr.'s Korkine and Zolotareff: *Sur les formes quadratiques*. [On the quadratic forms] ‡

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‡ Mathematische Annalen, V. VI, p. 367.

The formulae obtained give a mean for the study of various perfect forms which verify the inequality

$$\mathcal{M}(a_{ij}) > 2.$$

By making, for example, $n = 12$, one will have

$$m = 3 \quad \text{and} \quad p = 3.$$

By virtue of (34), one obtains

$$\rho = \frac{3}{4},$$

therefore, because of (35),

$$D = \frac{13}{16} \cdot \frac{1}{2^{12}},$$

and it follows that

$$\mathcal{M}(a_{ij}) = 2 \sqrt[12]{\frac{16}{13}} > 2.$$

All the extreme forms studied by Mr.'s Korkine and Zolotareff do not give a function $\mathcal{M}(a_{ij})$ of values which exceed 2.

On the quadratic perfect forms and on the domains which correspond to them.

43

We have seen in Number 29 that to the quaternary principal perfect form

$$\varphi = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4,$$

$$D = \tfrac{5}{2^4}$$

corresponds the domain R made up of forms

$$\rho_1 x_1^2 + \rho_2 x_2^2 + \rho_3 x_3^2 + \rho_4 x_4^2 + \rho_5 (x_1 - x_2)^2 + \rho_{[6]} (x_1 - x_3)^2 +$$

$$\rho_7(x_1-x_4)^2+\rho_8(x_2-x_3)^2+\rho_9(x_2-x_4)^2+\rho_{10}(x_3-x_4)^2.$$

All the perfect forms contiguous to the principal form φ are equivalent to the form

$$\varphi_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4,$$

$$D = \tfrac{1}{4}.$$

The corresponding domain R_1 is made up of forms

$$\rho_1x_1^2 + \rho_2x_2^2 + \rho_3x_3^2 + \rho_4x_4^2 + \rho_5(x_1 - x_3)^2 + \rho_6(x_1 - x_4)^2 + \rho_7(x_2 - x_3)^2 + \rho_8(x_2 - x_4)^2 + \rho_9(x_3 - x_4)^2 + \rho_{10}(x_1 + x_2 - x_3)^2 + \rho_{11}(x_1 + x_2 - x_4)^2 + \rho_{12}(x_1 + x_2 - x_3 - x_4)^2.$$

Let us examine the perfect forms contiguous to the perfect form φ_1.

We have demonstrated in Number 38 that all these forms are equivalent to the forms

1). $\varphi_1 - \rho x_1x_3$,
2). $\varphi_1 + \rho(x_1x_2 - \delta x_3x_4)$, where $\delta = 0$ or 1.

Let us examine three perfect forms

1). $\varphi_1 + \rho x_1x_2$, 2). $\varphi_1 - \rho x_1x_3$, 3). $\varphi_1 + \rho(x_1x_2 - x_3x_4)$.

1). By making $\rho = 1$ in the form $\varphi_1 + \rho x_1x_2$, one obtains the principal perfect form φ.

2). Let us notice that the form $\varphi_1 - \rho x_1x_3$ is equivalent to the form $\varphi_1 + \rho x_1x_2$.

In effect, the substitution

$$x_1 = -x_1', \; x_2 = x_3', \; x_3 = x_2', \; x_4 = x_1' + x_4'$$

does not change the form φ_1 and transforms the form x_1x_2 into the form $-x'_1x'_3$.

3). By making $\rho = 1$ in the form $\varphi_1 + \rho(x_1x_2 - x_3x_4)$, one obtains the form

$$x_1^2 + x_2^2, x_3^2 + x_4^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4$$

which is evidently equivalent to the perfect form φ_1.

One concludes that all the perfect forms contiguous to the perfect form φ_1 are equivalent to the forms φ and φ_1.

It follows that the set of all the quaternary perfect forms be divided into two classes represented by the perfect forms φ and φ_1.

The set (R) of domains corresponding to various quaternary perfect forms is made up of two classes, too, represented by the domains R and R_1.

On the perfect forms in five variables and on the domains which correspond to them.

44

We have determined two perfect forms in five variables

$$\varphi = x_1^2 + x_2^2 + \ldots + x_5^2 + x_1x_2 + x_1x_3 + \ldots + x_4x_5,\ D = \frac{6}{2^5},$$
$$\varphi_1 = x_1^2 + x_2^2 + \ldots + x_5^2 + x_1x_3 + x_1x_4 + \ldots + x_4x_5,\ D = \frac{4}{2^5}.$$

The corresponding domains R and R_1 will be composed of forms

$$R)\ \rho_1x_1^2 + \rho_2x_2^2 + \ldots + \rho_5x_5^2 + \rho_6(x_1 - x_2)^2 + \rho_7(x_1 - x_3)^2 + \ldots + \rho_{15}(x_4 - x_5)^2,$$
$$R_1)\ \rho_1x_1^2 + \rho_2x_2^2 + \ldots + \rho_5x_5^2 + \rho_6(x_1 - x_3)^2 + \ldots + \rho_{20}(x_1 + x_2 - x_4 - x_5)^2.$$

Examine the perfect forms contiguous to the perfect face φ_1. We have demonstrated in Number 38 that all these forms are equivalent to the forms

$$\begin{aligned} &1). \quad \varphi_1 - \rho x_1 x_3, \\ &2). \quad \varphi_1 - \rho(x_1x_2 - \delta x_3x_4 - \delta' x_3x_5 - \delta'' x_4x_5). \end{aligned} \tag{1}$$

where

$$\delta = 0 \text{ or } 1, \quad \delta' = 0 \text{ or } 1, \quad \delta'' = 0 \text{ or } 1.$$

In the second case one obtains 8 perfect forms. By permuting the variables x_3, x_4, x_5 one will replace the forms (1) by 4 forms; thus all the perfect forms contiguous to the perfect form φ_1 are equivalent to the 5 following forms:

1. $\varphi_1 + \rho x_1 x_2$,

2. $\varphi_1 + \rho x_1 x_3$,

3. $\varphi_1 + \rho(x_1x_2 - x_4x_5)$,

4. $\varphi_1 + \rho(x_1x_2 - x_3x_5 - x_4x_5)$,

5. $\varphi_1 + \rho(x_1x_2 - x_3x_4 - x_3x_5 - x_4x_5)$.

1). By making $\rho = 1$ in the perfect form $\varphi_1 + \rho x_1 x_2$, one obtains the perfect form φ.

2). We have seen in Number 42 that the perfect form $\varphi_1 - \rho x_1 x_3$ is determined by the value $\rho = 1$ of the parameter ρ in the case $n = 5$. One obtains the form

$$\varphi_1' = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_1x_4 + x_1x_5 + \ldots + x_4x_5 \tag{2}$$

which will be transformed with the help of the substitution

$$x_1 = -x_2^2,\ x_2 = x_1' - x_2',\ x_3 = x_3',\ x_4 = x_2 + x_4',\ x_5 = x_2' + x_5'$$

into a perfect form φ_1.

3). In the form $\varphi_1 + \rho(x_1x_2 - x_4x_5)$, one will put $\rho = 0$ and one will obtain the form

$$x_1^2 + x_2^2 + \ldots + x_5^2 + x_1x_2 + x_1x_3 + \ldots + x_3x_5$$

which is evidently equivalent to the form φ_1.

4). In the form $\varphi_1 + \rho(x_1x_2 - x_3x_5 - x_4x_5))$, one will put $\rho = 1$ and one will obtain the form

$$x_1^2 + x_2^2 + \ldots + x_5^2 + x_1x_2 + x_1x_3 + \ldots + x_2x_5$$

which is evident to the perfect form (2)

5). It remains only to determine the perfect form:

$$\varphi_1 + \rho(x_1x_2 - x_3x_4 - x_3x_5 - x_4x_5). \qquad (3)$$

By effecting the transformation with the help of the substitution

$$\begin{gathered} -x_1 + x_2 = x'_1, \\ x_1 + x_2 + x_3 + x_4 + x_5 = x'_2, \\ x_3 = x'_3, \ \ x_4 = x'_4, \ \ x_5 = x'_5 \end{gathered} \qquad (4)$$

of the form

$$2\varphi_1 + 2\rho(x_1x_2 - x_3x_4 - x_3x_5 - x_4x_5),$$

one obtains the form

$$\begin{aligned} & {x'_1}^2 + {x'_2}^2 + {x'_3}^2 + {x'_4}^2 + {x'_5}^2 + \\ & \frac{\rho}{2}\left[-{x'_1}^2 + {x'_2}^2 + {x'_3}^2 + {x'_4}^2 + {x'_5}^2 - 2x'_2x'_3 - 2x'_2x'_4 - \right. \\ & \qquad \left. 2x'_2x'_5 - 2x'_3x'_4 - 2x'_3x'_5 - 2x'_4x'_5\right]. \end{aligned} \qquad (5)$$

By virtue of (4) the integer variables $x'_1, x'_2, x'_3, x'_4, x'_5$ verify the congruence

$$x'_1 + x'_2 + x'_3 + x'_4 + x'_5 \equiv 0(\mathrm{mod} 2). \qquad (6)$$

By applying to the form (5) the method unveiled in Number 2, one will determine the value of the upper limit $R > 0$ of value $\frac{\rho}{2}$ with the help of equations

$$\begin{gathered}
\xi_1 - R\xi_1 7 = 0, \\
\xi_2 + R(\xi_2 - \xi_3 - \xi_4 - \xi_5) = 0, \\
\xi_3 + R(-\xi_2 + \xi_3 - \xi_4 - \xi_5) = 0, \\
\xi_4 + R(-\xi_2 - \xi_3 + \xi_4 - \xi_5) = 0, \\
\xi_5 + R(-\xi_2 - \xi_3 - \xi_4 + \xi_5) = 0.
\end{gathered}$$

It results in that

$$\xi_2 = \xi_3 = \xi_4 = \xi_5,$$

and one obtains the equations

$$\xi_1(1 - R) = 0 \quad \text{and} \quad \xi_2(1 - 2R) = 0,$$

thus

$$R = \frac{1}{2}.$$

By declaring

$$x_1' = 0,\ x_2' = 1,\ x_3' = 1,\ x_4' = 1, x_5' = 1, \tag{7}$$

one will satisfy the condition (6) and one will have the value $4 - 4\rho$ of the form (5).

By making

$$4 - 4\rho = 2,$$

one obtains

$$\rho = \frac{1}{2}.$$

It follows that the positive quadratic form

$$x_1'^2 + x_2'^2 + \ldots + x_5'^2 + \\ \frac{1}{4}\left[-x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2 + x_5'^2 - 2x_2'x_3' - \ldots - 2x_4'x_5'\right] \tag{8}$$

will have a value 2 corresponding to the system (7).

By virtue of that which has been discussed in Number 23, the smallest value of the form (8) will correspond to a system $(l_1, l_2, \ldots, l_5)$ verifying the inequality

$$l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 \leq 2 \cdot \frac{R}{R - \frac{1}{4}} \text{ where } R = \frac{1}{2}.$$

One obtains the inequality

$$l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 \leq 4.$$

It is easy to demonstrate that the system (7) is the only one verifying this inequality on condition (6), the systems which verify the inequality

$$-{x'_1}^2 + {x'_2}^2 + {x'_3}^2 + {x'_4}^2 + {x'_5}^2 - 2x'_2x'_3 -$$
$$2x'_2x'_4 - 2x'_2x'_5 - 2x'_3x'_4 - 2x'_3x'_5 - 2x'_4x'_5 \geq 0$$

being excluded. By making $\rho = \frac{1}{2}$ in the form (3), one obtains the perfect form

$$\varphi_2 = x'_1 + x'_2 + \ldots + x'_5 + \frac{1}{2}x_1x_2 + x_1x_3 + \ldots$$
$$+ x_2x_5 + \frac{1}{2}x_3x_4 + \frac{1}{2}x_3x_5 + \frac{1}{2}x_4x_5,$$
$$D = \left(\frac{3}{2}\right)^4 \frac{1}{2^5}.$$

The corresponding domain R^2 is composed of forms

$$\rho_1x_1^2 + \rho_2x_2^2 + \ldots + \rho_5x_5^2 + \rho_6(x_1 - x_3)^2 + \ldots + \rho_{11}(x_2 - x_5)^2$$
$$+ \rho_{12}(x_1 + x_2 - x_3 - x_4)^2 + \rho_{13}(x_1 + x_2 - x_3 - x_5)^2$$
$$+ \rho_{14}(x_1 + x_2 - x_4 - x_5)^2 + \rho_{15}(-x_1 - x_2 + x_3 + x_4 + x_5)^2.$$

The number of parameter $\rho_1, \rho_2, \ldots, \rho_{15}$ being equal to the number of dimensions of the domain R_2, one will determine without trouble 15 inequalities which define the domain R_2.

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We have demonstrated that all the perfect forms contiguous to the perfect form φ_1 are equivalent to the perfect forms φ, φ_1 and φ_2.

Choose the perfect forms contiguous to the perfect form φ_2.

To this effect let us notice, in the first place, that the perfect form φ_1 is contiguous to the perfect form φ_2, then observe that all the perfect forms contiguous to the form φ_2 are equivalent.

To demonstrate this, examine all the faces in 14 dimensions of the domain R_2.

The domain R_2 is characterised by 15 quadratic forms

$$\begin{cases} x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, \\ (x_1 - x_3)^2, (x_1 - x_4)^2, (x_1 - x_5)^2, \\ (x_2 - x_3)^2, (x_2 - x_4^2), (x_2 - x_5)^2, \\ (x_1 + x_2 - x_3 - x_4)^2, (x_1 + x_2 - x_3 - x_5)^2, \\ (x_1 + x_2 - x_4 - x_5)^2, (-x_1 - x_2 + x_3 + x_4 + x_5)^2. \end{cases} \tag{9}$$

Each face in 14 domains of the domain R_2 possesses 14 of these, and the form which remains can be called form opposite to the face.

One concludes that each face is well determined by the opposite face.

For the perfect forms contiguous to the perfect form φ_2 to be equivalent, it is necessary and sufficient that all the faces of the domain R_2 could be transformed one to one with the help of substitutions which do not change the domain R_2.

It would be easy to write all these substitutions, but one will proceed in another way, more speedy.

Let us observe that the face P belonging to the domain R_1 and R_2 is characterised by all the forms (9), the form $(-x_1 - x_2 + x_3 + x_4 + x_5)^2$ being excluded.

With the aid of substitution associated with the substitution (4), one will replace the forms (9) by the forms:

$$\begin{cases} (x'_1 \pm x'_2)^2, (x'_1 \pm x'_3)^2, (x'_1 \pm x'_4)^2, (x'_1 \pm x'_5)^2, \\ (x'_2 + x'_3)^2, (x'_2 + x'_4)^2, (x'_2 + x'_5)^2, (x'_3 + x'_4)^2, (x'_3 + x_{[}5]')^2, \\ (x'_4 + x'_5)^2, (x'_2 + x'_3 + x'_4 + x'_5)^2. \end{cases} \tag{10}$$

By changing the sign of x'_1 and by permuting the variables x'_2, x'_3, x'_4, x'_5, one will transform into itself the forms (10), and the form $(x'_2 + x'_3 + x'_4 + x'_5)^2$ will not change.

To each similar substitution corresponds a substitution which transforms into itself the domain R_2 and the face P of the domain R_2, and does not change the form $(-x_1 - x_2 + x_3 + x_4 + x_5)^2$.

By changing the sign of x'_1 and by permuting x'_2, x'_3, x'_4, x'_5, one will transform the form $(x'_1 + x'_2)^2$ into forms

$$(x'_1 \pm x'_2)^2, (x'_1 \pm x'_3)^2, (x'_1 \pm x'_4)^2, (x'_1 \pm x'_5)^2$$

and one will transform the form $(x'_2 + x'_3)^2$ into forms

$$(x'_2 + x'_3)^2, (x'_2 + x'_4)^2, (x'_2 + x'_5)^2,$$

$$(x'_3 + x'_4)^2, (x'_3 + x'_{[5]})^2, (x'_4 + x'_5)^2.$$

Thus only the forms

$$(x'_1 + x'_2)^2,\ (x'_2 + x'_3)^2,\ (x'_2 + x'_3 + x'_4 + x'_5)^2 \tag{11}$$

remain to examine.

By returning to the forms (9), one obtains the forms corresponding to the forms (11).

$$x_2^2, x_3^2, (-x_1 - x_2 + x_3 + x_4 + x_5)^2. \tag{12}$$

It is demonstrated that all the forms (9) can be transformed into forms (12) with the help of substitutions which do not change the domain R_2.

With the help of substitutions

$$\begin{aligned}
x_1 &= x'_2 - x'_5,\\
x_2 &= x'_3,\\
x_3 &= x'_2,\\
x_4 &= x'_1 + x'_2 - x'_4 - x'_5,\\
x_5 &= -x'_1 + x'_3
\end{aligned}$$

and

$$\begin{aligned}
x_1 &= x'_1,\\
x_2 &= -x'_1 - x'_2 + x'_3 + x'_4 + x'_5,\\
x_3 &= x'_3,\\
x_4 &= x'_4,\\
x_5 &= x'_5,
\end{aligned}$$

one will transform the domain R_2 into itself, and the form x_2^2 will be transformed into forms x_3^2 and $(-x'_1 - x'_2 + x'_3 + x'_4 + x'_5)^2$.

We have demonstrated that all the forms of the domain R_2 are equivalent. It results in, from that we have seen, that all the perfect forms contiguous to the perfect form φ_2 are equivalent to the perfect form φ_1.

One concludes that all the perfect forms in five variables constitute three different classes represented by the perfect forms φ, φ_1 and φ_2.

The set of domains (R) can be divided into three classes also, represented by the domains R, R_1, and R_2.

End of the first Mémoire.